Analysis of Partial-Wave Dispersion Relations*

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Questions of existence and uniqueness of solutions of partial-wave dispersion relations are studied, with particular attention to the N/D method. The interaction, assumed to be given, is represented by (i) the strengths and locations of unphysical singularities and (ii) the inelastic partial-wave cross section. A generalization of the N/D method to include part (ii) of the interaction leads to a nonsingular integral equation for ImD. This equation is amenable to the Fredholm theory only if there is a correlation between items (i) and (ii) of the interaction, and only if the increase of inelastic processes at high energies is not too rapid. Certain Cauchy integrals associated with (i) and (ii) must be nonzero at threshold if there is to exist a solution with the normal threshold momentum dependence. Thus, there is no solution for any model in which (i) is constructed from a few partial waves in the two crossed channels. For certain interactions the real part of the phase shift approaches a multiple of π at large energy, just as in potential scattering. The Castillejo-Dalitz-Dyson (CDD) ambiguity is analyzed in some detail. A uniqueness theorem is proved which asserts that if a solution of a particular type exists, it is the only solution of the problem within the class usually considered. Thus the CDD ambiguity is partially bypassed. In certain cases the unique solution is found by the ordinary N/D method without subtractions. Some useful results on principal value integrals are obtained. The discussion is carried out for the example of pion-nucleon scattering in the complex plane of w, the center-of-mass energy. The behavior of the amplitude near $w=\pm(M-m)$ is derived from crossing symmetry.

I. INTRODUCTION

RECENTLY, the partial-wave dispersion relations have played an important part in discussions of the strong interactions.[1] If the discontinuity of the amplitude over the unphysical cut (the "left" cut) is somehow known approximately, the dispersion relation amounts to a singular integral equation for the amplitude. This equation is replaced by a nonsingular one through the N/D method. A solution of the latter will sometimes provide a solution of the former. This role of the partial-wave dispersion relation has occasionally been compared to that of the Schrödinger equation in nonrelativistic quantum theory. However, the analogy is imperfect. For one thing, the jump over the left cut, which now takes the part of the interaction Hamiltonian, must be specified for each partial wave separately. But more important, the existence and uniqueness theorems that testify to the reasonable nature of the Schrödinger problem are almost completely lacking. Solutions can fail to exist if the N/D solution involves a spurious "ghost" pole not represented in the dispersion relation. Furthermore, the work of Castillejo, Dalitz, and Dyson[2] seems to show that a solution is never unique (however, compare our Sec. V).

In spite of these essential mathematical differences it is still desirable to improve the analogy with the Schrödinger problem, at least in a practical sense. That is to say, we should like to know before explicit calculation whether a particular interaction will lead to a solution satisfying all known general requirements. Also, we should like to claim uniqueness of a solution, if possible, and perhaps to estimate some of its qualitative features without extensive computations. The purpose of the present paper is to see how nearly these aims can be realized. Especially, we try to make the N/D method more useful and more flexible through a better understanding of the mathematical questions involved.

The problem is important not only in the original Chew-Mandelstam theory, which has had only limited success, but also in the more ambitious schemes suggested by Mandelstam,[1] Ter-Martirosyan,[3] Zimmermann,[4] Chew and Frautschi,[5] Wilson,[6] and others. In these theories it seems necessary to solve at least the S-wave dispersion relation. As a result of bad asymptotic behavior of the approximate interaction term, the original theory is not strictly consistent. Therefore, it seems necessary to consider the generalizations. In that case the approximation of purely elastic scattering must be abandoned, and the N/D method appropriately modified. According to Chew and Frautschi,[7] the partial-wave dispersion relation is to be solved with the following two items regarded as given information: (i) a function f^U which represents the usual contribution of the unphysical cuts; (ii) a function f^I which represents inelastic effects. f^I is determined by the partial-wave inelastic cross section. Ball and Frazer[8] have

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960), and papers of Mandelstam cited therein. See also the lectures of G. F. Chew in *Relations de dispersion et particules elementaires* (Hermann et Cie, Paris, and John Wiley & Sons, Inc., New York, 1960), and recent papers by authors too numerous to mention.

[2] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956); hereafter referred to as CDD.

[3] K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. (U.S.S.R.) **39**, 827 (1960) [translation: Soviet Phys.—JETP **12**, 575 (1961)].

[4] W. Zimmermann, Nuovo Cimento **21**, 36 (1961).

[5] G. F. Chew and S. C. Frautschi, Phys. Rev. **123**, 1478 (1961). See also reference 7, and G. F. Chew, Lawrence Radiation Laboratory Report UCRL-9515 (unpublished).

[6] K. Wilson (unpublished).

[7] G. F. Chew and S. Frautschi, Phys. Rev. **124**, 264 (1961).

[8] J. S. Ball and W. R. Frazer, Phys. Rev. Letters **7**, 204 (1961).

shown that item (ii) can be qualitatively important. In the approximation in which (i) is neglected they showed that the problem has a very simple solution that can be stated in closed form. Ball and Frazer also noted that both (i) and (ii) could be retained in a modified N/D procedure based on a definition of D due to Chew and Frautschi.[9] However, the resulting linear integral equation had a singular form involving repeated principal value integrations.

After establishing notation and stating the problem precisely in Sec. II, we discuss this generalized N/D method in Sec. III. It turns out that a careful change of integration order puts the equation in nonsingular form. Thus a nonsingular equation for $\mathrm{Im}D$ ($\propto \mathrm{Re}N$) is derived from the singular equation for N. The advantage of beginning with the N equation rather than the D equation is that eventually only integrations over physical energies are involved. In the elastic case the corresponding equation has been employed by Uretsky.[10] Our equation has nearly the same form as Uretsky's, except that the absorption factor η enters in a curious way. Here η is $\exp(-2\,\mathrm{Im}\delta)$, δ being the complex phase shift. Although the equation has the Fredholm form, the Fredholm theory applies only if the kernel and inhomogeneous term are integrable in the square (L^2). This point is investigated in some detail, since the square-integrability is in doubt if η vanishes at infinity. Sufficient conditions for an L^2 kernel are derived, and some restrictions on the interaction terms are discovered. It is found that f^U and f^I cannot be chosen independently, in general. Their asymptotic behaviors must be precisely matched. In the course of this discussion we prove that in certain cases the real part of the phase shift must approach an integral multiple of π at infinity, just as in potential scattering. Section III concludes with a derivation of the N equation which is simpler than the obvious one. It also eliminates an unnecessary assumption and forms the starting point for Sec. VI.

Section IV is concerned with conditions necessary for the existence of solutions. We bring up a point that has so far not received adequate attention, viz., the requirement that the amplitude have the expected threshold zeros. We treat the special case of spin 0-spin $\frac{1}{2}$ scattering; it is particularly interesting in this respect. (In fact, throughout the paper we consider just this example.) A simple argument shows that if the interaction term by itself has the threshold zeros, then there is no solution of the partial-wave dispersion relation with correct zeros. It follows that any interaction derived from just a few partial waves in the two crossed channels cannot lead to a satisfactory solution. This situation is made comprehensible within the framework of the Cini-Fubini representation[11] by showing that a finite number of partial waves in the direct channel gives a set of poles of ascending order at the origin in the energy plane. These pole terms do not possess the threshold zeros, so if they are included a solution becomes possible. We reformulate the N/D technique so that the threshold conditions are automatically satisfied at the expense of introducing poles at the origin.

In Sec. V we turn to the uniqueness question. We find that in certain circumstances the ambiguity of Castillejo *et al.*[2] may essentially disappear. In fact, if there is a solution of a particular type, it is the only solution of the problem within the class of solutions usually considered. Thus, the possibility of uniqueness depends on the nature of the interaction. The threshold conditions help to pin down the solution, and therefore are analogous to boundary conditions on the wave function in the Schrödinger theory.

The condition for uniqueness depends on the orbital angular momentum l, and becomes less stringent as l increases. For $l\geqslant 1$, if a solution exists, it is quite likely to be unique. If $l\geqslant 1$ and if the equation for $\mathrm{Im}D$ without subtractions has square-integrable kernel and inhomogeneous term, the amplitude constructed from its solution is a unique solution of the partial-wave dispersion relations provided it has no ghosts.

The topic of Sec. VI is the incorporation of the CDD ambiguity in the N equation with inelastic effects allowed. In the elastic case, Chew,[1] Chew and Frautschi,[7] and Gell-Mann and Zachariasen[12] have shown how the so-called "CDD poles" enter the N/D scheme, and have associated the corresponding free parameters with the masses and widths of unstable elementary particles. When inelastic processes are included some extra care is necessary in integrating over singularities, but the result is simple. Only the inhomogeneous term of the N equation is altered.

We find that CDD poles correspond to zeros of the amplitude only below the inelastic threshold. Above the threshold the amplitude has the value $i(1-\eta)/2k$ at a CDD pole, where k is the center-of-mass momentum.

In Sec. VI it is proved that any amplitude has a particularly elegant N/D representation in which D is a so-called Herglotz function (generalized Wigner R function). This generalizes the work of Castillejo *et al.* to the cases in which the unphysical singularities may include branch points as well as poles. The Herglotz D is not necessarily identical to the usual D, but an exploration of the connection between the two throws light on the mathematical situation. Part of the work of Sec. VI depends on the Herglotz D.

In Appendixes A through D we prove some necessary theorems on the behavior of principal value integrals.

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[9] G. F. Chew and S. Frautschi, Lawrence Radiation Laboratory Report UCRL-9685 (unpublished). This is evidently a preliminary version of reference 7. Our definition of D agrees with Eq. (5) of UCRL-9685, while the corresponding equation of the published paper is in accord with some work of Froissart (cf., reference 28).

[10] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).

[11] M. Cini and S. Fubini, Ann. Phys. (N. Y.) **3**, 352 (1960).

[12] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 1961).

These results might be generally useful in further work on dispersion relations. Appendix E is concerned with the behavior of the pion-nucleon partial-wave amplitude at the points $w=\pm(M-m)$.

II. PARTIAL-WAVE DISPERSION RELATIONS

We study the elastic scattering amplitude for a problem in which the two incoming particles have masses (respective spins) m (0) and M ($\frac{1}{2}$). In other cases our discussion requires only small changes. In part of the work, especially Appendix E, we specialize to pion-nucleon scattering.

Let w and k be the barycentric total energy and magnitude of three-momentum. Then $4sk^2=[s-(M+m)^2]\times[s-(M-m)^2]$, where $s=w^2$. The scattering in a definite isotopic spin state (index suppressed) and parity-angular momentum state with $J=l\pm\frac{1}{2}$ is described by the amplitude

$$f_{l\pm}(w)=\{\eta_{l\pm}(w)\exp[2i\,\mathrm{Re}\delta_{l\pm}(w)]-1\}/2ik(w), \tag{II.1}$$

where $\delta_{l\pm}$ is the complex phase shift and $\eta_{l\pm}=\exp(-2\,\mathrm{Im}\delta_{l\pm})$; $0\leqslant\eta_{l\pm}\leqslant 1$. The unitarity condition is

$$\mathrm{Im}f_{l\pm}=k|f_{l\pm}|^2+(1-\eta_{l\pm}{}^2)/4k, \tag{II.2}$$

for $w\geqslant w_0=M+m$. We adopt the convention $\delta_{l\pm}(w_0)=0$.

In order to avoid trouble from kinematical branch points, it is best to consider together both amplitudes having the same J, as shown by Frazer and Fulco.[13] We define a function $f_{l\pm}(z)$, analytic in the cut-z plane. The cuts are as follows: (i) the physical cut, hereafter called P, consisting of two parts of the real axis $(-\infty<z<-w_0, w_0<z<\infty)$; (ii) unphysical cuts U elsewhere in the plane. There may be isolated poles as well. The cuts U as given by the Mandelstam representation are described in reference 13. However, the partial-wave dispersion relations may be valid more generally than the Mandelstam representation.[14] In any event, our work is independent of the details of these cuts, provided they do not intersect the physical cut. Now $f_{l\pm}(z)$ satisfies the MacDowell[13,15] relation

$$f_{l\pm}(z)=-f_{(l\pm1)\mp}(-z). \tag{II.3}$$

Thus, $f_{l+}(w+i0)$, $w>w_0$, is the scattering amplitude for $l=J-\frac{1}{2}$, while $-f_{l+}(-w-i0)$, $w>w_0$, is the amplitude for $l=J+\frac{1}{2}$.

It is useful to define some abbreviations. We write $f(z)\equiv f_{l+}(z)$ and $f(w)\equiv f(w+i0)$, where w is understood to be a real point on P. Since we assume the Riemann-Schwarz condition $f(z^*)=f^*(z)$, it follows from (II.1) and (II.3) that

$$f(w)=[\eta(w)e^{2i\delta(w)}-1]/2i\kappa(w), \tag{II.4}$$

where

$$\begin{aligned}\eta(w)&=\eta_{l+}(w), & w&>w_0\\ &=\eta_{(l+1)-}(-w), & w&<-w_0\end{aligned} \tag{II.5}$$

$$\begin{aligned}\delta(w)&=\mathrm{Re}\delta_{l+}(w), & w&>w_0\\ &=-\mathrm{Re}\delta_{(l+1)-}(-w), & w&<-w_0\end{aligned} \tag{II.6}$$

and $\kappa(w)=k(|w|)$.

The dispersion relation is taken to be[16,17]

$$f(z)=f^U(z)+\frac{1}{\pi}\int_P dw\,\frac{\mathrm{Im}f(w)}{w-z}, \tag{II.7}$$

where

$$f^U(z)=\frac{1}{\pi}\int_U dz'\,\frac{\Delta f(z')}{z'-z}. \tag{II.8}$$

Here $\Delta f(z)$ is $1/2i$ times the discontinuity over the cut U. Δf may contain delta functions to account for possible poles of f. The convergence of the integral over P is assured by unitarity. The convergence of (II.8) and the vanishing of the integral over the contour at infinity presumably follow from arguments of the Pomeranchuk type.[18,19] Roughly speaking, these arguments show that the amplitude should have the same asymptotic behavior in all directions in the complex plane.

It is instructive to rewrite (II.7) in various ways. To emphasize the presence of two orbital states we use (II.3) and find

$$f_{l+}(z)=f_{l+}{}^U(z)+\frac{1}{\pi}\int_{w_0}^{\infty}dw\left[\frac{\mathrm{Im}f_{l+}(w)}{w-z}-\frac{\mathrm{Im}f_{(l+1)-}(w)}{w+z}\right]. \tag{II.9}$$

Incorporation of (II.2) shows that the density function

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[13] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).

[14] J. G. Taylor, Nuovo Cimento **22**, 92 (1961); N. Nakanishi, Phys. Rev. **126**, 1225 (1962).

[15] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

[16] To ensure the correctness of the usual formula $(w-z-i\epsilon)^{-1}=P((w-z)^{-1})+i\pi\delta(w-z)$ in taking the limit $z\to w$ it is necessary to make some statement about the continuity properties of the density function $\mathrm{Im}f(w)$. It is sufficient to impose the Hölder condition; cf. reference 17, Secs. 16 and 17. A function $\varphi(x)$ is said to satisfy an H (Hölder) condition on a finite interval L if $|\varphi(x_1)-\varphi(x_2)|\leqslant A|x_1-x_2|^\mu$, $A>0$, $0<\mu\leqslant 1$, for any two points x_1, x_2 of L. We write "φ belongs to H" or "$\varphi\in H$." We assume explicitly that $\eta\in H$ and $\delta\in H$; therefore $\mathrm{Im}f(w)\in H$. The H condition is more appropriate for physics than the stronger requirement that the functions have a derivative. The latter condition fails at least at two-body, S-wave thresholds; e.g., there is a cusp phenomenon at the π-Σ threshold in S-wave π-Λ scattering, assuming even Λ-Σ parity. In the case of an infinite interval L, the H condition is supposed to hold on any finite subinterval.

[17] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[18] For the present case we know of no complete proof of a Pomeranchuk-type theorem {cf., I. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. (U.S.S.R.) **34**, 725 (1958) [translation: Soviet Physics—JETP, **7**, 499 (1958)]}. A method that might be adapted to handle the problem is given by Weinberg in reference 19. It is probably necessary to assume that the amplitude is uniformly bounded by a polynomial and that the density functions have at most a finite number of zeros.

[19] S. Weinberg, Phys. Rev. **124**, 2049 (1961).

is positive on P.

$$f(z)=f^U(z)+\frac{1}{\pi}\int_P dw\,\frac{\kappa(w)|f(w)|^2}{w-z}+\frac{1}{4\pi}\int_I dw\,\frac{[1-\eta^2(w)]}{\kappa(w)(w-z)}. \tag{II.10}$$

I is the inelastic region $|w|>w_{\rm in}$, $w_{\rm in}$ being the threshold for inelastic events. Finally, we have a form that is useful in Sec. III.

$$f(z)=f^U(z)+f^I(z)+\frac{1}{\pi}\int_P dw\,\frac{\eta(w)\sin^2\delta(w)}{\kappa(w)(w-z)}. \tag{II.11}$$

where

$$f^I(z)=\frac{1}{\pi}\int_I dw\,\frac{1-\eta(w)}{2\kappa(w)(w-z)}. \tag{II.12}$$

The real part of the function $B=f^U+f^I$ appears in the N equation.

When f^U and η (or equivalently f^U and f^I) are regarded as given functions (II.10) becomes a singular, nonlinear equation for $f(w)$ in the limit $z\to w+i0$. To find a corresponding nonsingular and linear equation we use the N/D representation of Chew and Frautschi[9] in which $1/D$ has as its phase the function $\delta(w)$ defined by (II.6). N is analytic in the plane with cuts U and I, while D satisfies $D(z^*)=D^*(z)$ and is analytic in the plane cut by P. N and D may have poles at infinity, but it is understood that D has no poles superimposed on P (however, see Sec. VI). It is not clear that any amplitude satisfying (II.10) has such a representation. The possibility of the representation can be proved if $\delta(w)$ is bounded on P. From here on it will be understood that we are considering only the class of solutions for which $\delta(w)$ is bounded. From the δ corresponding to a given amplitude f construct the function $\mathfrak{D}$.

$$\mathfrak{D}(z)=\exp\left(-\frac{z}{\pi}\int_P\frac{dw\,\delta(w)}{w(w-z)}\right). \tag{II.13}$$

We have $\mathfrak{D}(z^*)=\mathfrak{D}^*(z)$, and $\mathfrak{D}(z)$ is analytic in any finite region of the plane cut by P. By (II.4), $\mathfrak{N}\equiv f\mathfrak{D}$ is real in the elastic region $w_0\leqslant|w|\leqslant w_{\rm in}$. Since $\mathfrak{N}(z^*)=\mathfrak{N}^*(z)$, the jump of $\mathfrak{N}$ over P is proportional to its imaginary part and, therefore, zero in the elastic region. Thus, the example $f=\mathfrak{N}/\mathfrak{D}$ shows that any f has an N/D representation as described. Of course, the decomposition into N and D is not unique. N and D may each be multiplied by a common polynomial with real coefficients. Furthermore, any D that is $O(|z|^n)$[20] for some n can be written $D=\Phi\mathfrak{D}$, where Φ is a polynomial.

[20] We use the symbols O, o, and $\sim$ in the customary way. $f(x)=O[g(x)]$ means that $|f(x)|\leqslant Mg(x)$ for some fixed M and all x sufficiently close to a given limit. By $f(x)=o[g(x)]$ we mean that $f(x)/g(x)\to 0$ as x tends to a given limit, while $f(x)\sim g(x)$ is to mean that $f(x)/g(x)\to 1$. Thus, $D(z)=O(|z|^n)$ indicates that $|D|\leqslant M|z|^n$ for all $|z|>r$.

To prove this, note that $D/\mathfrak{D}$ is a function devoid of singularities in any finite region. Also, $D/\mathfrak{D}=O(|z|^m)$ for some m, by the theorem of Appendix A. Therefore, $D/\mathfrak{D}$ is a polynomial.

When the full notation is substituted for the compact writing of (III.13), we have

$$\mathfrak{D}(z)=\mathfrak{D}(z;\,\mathrm{Re}\delta_{l+},\,\mathrm{Re}\delta_{(l+1)-})$$
$$=\exp\left(-\frac{z}{\pi}\int_{w_0}^{\infty}\frac{dw\,\mathrm{Re}\delta_{l+}(w)}{w(w-z)}+\frac{z}{\pi}\int_{w_0}^{\infty}\frac{dw\,\mathrm{Re}\delta_{(l+1)-}(w)}{w(w+z)}\right). \tag{II.14}$$

The behavior at infinity of $\mathfrak{D}$ is related to that of δ. If $\delta(w)$ approaches a constant as $|w|\to\infty$ we have the theorem of Appendix A:

$$\mathfrak{D}(z)=O(|z|^{p+\epsilon}) \tag{A6}$$

for all $\epsilon>0$, where $\pi p=\delta_{l+}(\infty)+\delta_{(l+1)-}(\infty)$. See, also, Eq. (A7). If $\delta(w)$ oscillates at infinity, $\mathfrak{D}$ is still bounded by a power of $|z|$; cf., Eq. (A9). With (A6) we can deal with the question of subtractions in the Cauchy integral representations of $\mathfrak{N}$ and $\mathfrak{D}$. Suppose that δ approaches a constant. If $p<1$, then ϵ can be chosen small enough to show that $\mathfrak{D}(z)z^{-1}=O(|z|^{-\delta})$, $\delta>0$. In that case, $\mathfrak{D}$ satisfies a dispersion relation with one subtraction. If $p>1$, additional subtractions are needed; the subtraction terms involve arbitrary constants which represent the CDD ambiguity. The case $p=1$ also involves a CDD ambiguity, in general, although the Cauchy integral may converge with one subtraction. See Secs. VI and VII for a clarification of this point.

When $p<1$, the one necessary subtraction does not imply a lack of uniqueness in the solution of (II.10), since without restricting the amplitude $\mathfrak{D}$ may be given any desired value (at a point where it is real) through multiplication by a real constant. On the other hand, we cannot immediately rule out the possibility of a subtraction in the dispersion relation for $\mathfrak{N}$. None would be necessary if f had the unitarity bound in all complex directions uniformly [i.e., $f(z)=O(|z|^{-1})$]. But that is a stronger statement than is essential for (II.7), and it is actually unnecessary for the formulation of the N equation without arbitrary constants, as is shown in the next section. However, in the next section we first assume $f(z)=O(|z|^{-1})$ in order to derive the N equation by the obvious route of eliminating D between the two dispersion relations for N and D. Later we give a method which eliminates the unnecessary assumption. The second method is actually simpler, even if less obvious.

III. N/D METHOD ALLOWING INELASTIC PROCESSES

Equation (II.4) implies $\exp(2i\delta)=D^*/D$ and $N=(\eta D^*-D)/2i\kappa$, where the functions are all evaluated in the limit $z \to w+i0$, w on P. It follows that

$$\mathrm{Re}N = -(1+\eta)\,\mathrm{Im}D/2\kappa, \quad \text{(III.1)}$$

$$\mathrm{Im}N = (1-\eta)\,\mathrm{Re}D/2\kappa. \quad \text{(III.2)}$$

Now any $f=O(|z|^{-1})$ in the class for which δ approaches a constant and $p<1$ has a representation N/D such that N and D satisfy the following dispersion relations:

$$N(z)=\frac{1}{\pi}\int_U dz'\,\frac{\Delta f(z')D(z')}{z'-z} +\frac{1}{\pi}\int_I dw\left(\frac{1-\eta}{2\kappa}\right)\frac{\mathrm{Re}D(w)}{w-z}, \quad \text{(III.3)}$$

$$D(z)=1+\frac{z}{\pi}\int_P \frac{dw}{w}\left(\frac{-2\kappa}{1+\eta}\right)\frac{\mathrm{Re}N(w)}{w-z}. \quad \text{(III.4)}$$

Substitute (III.4) in (III.3), and take the limit $z\to w+i0$.[21] The real part of the resulting equation is a singular integral equation for $\mathrm{Re}N(w)$, with w on P. A reversal of the order of integrations is helpful. In the U integral the reversal is easily justified. We find

$$f^U(w)+\frac{1}{\pi}\int_P dw'\,\frac{\kappa'}{w'}\left[\frac{w'f^U(w')-wf^U(w)}{w'-w}\right]n(w), \quad \text{(III.5)}$$

where $-\kappa n=\mathrm{Im}D$.

Since $f^U(w)$ is differentiable, the integrand is well defined at $w=w'$. In the I integral we have repeated principal value integrations which may be interchanged by means of the Poincaré-Bertrand formula[22]

$$P\int_L \frac{dx}{x-t}P\int_L \frac{\varphi(x,y)dy}{y-x} = -\pi^2\varphi(t,t)+\int_L dy\,P\int_L \frac{\varphi(x,y)dx}{(x-t)(y-x)}. \quad \text{(III.6)}$$

The integral over y on the right of (III.6) exists in the ordinary Riemann sense because of cancellation of one pole by the other. When (III.6) is applied to the I integral of (III.3), an expression just like (III.5) appears, but with f^U replaced by $\mathrm{Re}f^I$. According to Appendix B, the expression $[w'\,\mathrm{Re}f^I(w')-w\,\mathrm{Re}f^I(w)]/(w'-w)$ is integrable at $w=w'$. An additional contribution, arising from the $\varphi(t,t)$ term in (III.6), combines in an interesting way with $\mathrm{Re}N$ on the left side of (III.3). Collecting terms, we have

$$\eta(w)n(w)=\mathrm{Re}B(w)+\frac{1}{\pi}\int_P dw'\left(\frac{\kappa'}{w'}\right) \times\left[\frac{w\,\mathrm{Re}B(w)-w'\,\mathrm{Re}B(w')}{w-w'}\right]n(w'), \quad \text{(III.7)}$$

where

$$\mathrm{Re}B=\mathrm{Re}(f^U+f^I)=f^U+\mathrm{Re}f^I.$$

A solution of (III.7) is used with (III.3) and (III.4) to calculate $f=N/D$. In practice, it is perhaps better to calculate N by means of (III.19) in which case we have

$$f(z)=B(z)+\frac{1}{\pi D(z)}\int_P dw\,\frac{\kappa n(w)\,\mathrm{Re}B(w)}{w-z}. \quad \text{(III.8)}$$

By showing that (III.1) and (III.2) are satisfied it is easy to see that f satisfies (II.10) provided (a) D has no zero that cannot be called a bound-state zero, and (b) f is such that its Cauchy integral over the contour at infinity vanishes and its integral over U converges [see the remarks following (II.7)]. If (a) is not satisfied, there is no solution of (III.10) within the class for which $p<1$, since the equations we have developed are necessary conditions on solutions in that class. The exact number of zeros of D can be found by determining the degree of the polynomial $\Phi=D/\mathfrak{D}$, where $\mathfrak{D}$ is calculated from (II.13) and the phase δ of D^{-1}.[23] The results of Appendixes A and D will help to reduce the amount of computation necessary in finding the degree of Φ.

The substitution $w=w_0/s$ puts (III.7) into the standard form (III.9) of a Fredholm equation of the second kind.

$$x(s)=y(s)+\int_{-1}^{1}K(s,t)x(t)dt,$$

$$x=w\eta^{1/2}n;\quad y=w\eta^{-1/2}\,\mathrm{Re}B, \quad \text{(III.9)}$$

$$\pi w_0K=\left(\frac{\kappa'}{w'}\right)\frac{ww'}{(\eta\eta')^{1/2}}\left[\frac{w\,\mathrm{Re}B(w)-w'\,\mathrm{Re}B(w')}{w-w'}\right].$$

The Fredholm theory applies if $y\in L^2$, $K\in L^2$, i.e.,

$$\int_{-1}^{1}y^2(s)ds<\infty,\quad \int_{-1}^{1}\int_{-1}^{1}K^2(s,t)dsdt<\infty. \quad \text{(III.10)}$$

In this case, (III.9) has exactly one solution in the class L^2, unless the Fredholm determinant should happen to vanish.[24] Since η appears in the denominator of K, and η presumably vanishes at infinity (dominance of inelastic processes at high energies), the conditions (III.10) are very much in question. Only the point at infinity in the w, w' plane is potentially dangerous. Since $2\kappa/w\sim 1$, conditions (III.10) are then equiva-

[21] Here, one must know that $\mathrm{Re}D$ and $\mathrm{Im}D$ satisfy the H condition. See Appendix B.

[22] Conditions for the applicability of (III.6) are treated in Appendix C.

[23] This is the essential idea of a note by H. Sugawara and A. Kanazawa, Phys. Rev. **126**, 2251 (1962). We wish to thank the authors for a pre-publication copy of this work.

[24] As far as we know, a vanishing Fredholm determinant would have no physical significance and could occur only as a result of some approximation in constructing the interaction.

lent to

$$\int_P dw\,[\mathrm{Re}B(w)]^2/\eta(w)<\infty\,, \qquad \text{(III.11a)}$$

$$\int_P\int_P dwdw'\frac{1}{\eta(w)\eta(w')}\times\left[\frac{w\,\mathrm{Re}B(w)-w'\,\mathrm{Re}B(w')}{w-w'}\right]^2<\infty\,. \qquad \text{(III.11b)}$$

Although the mathematical question of the convergence of these integrals may be quite different in different cases, it is nonetheless instructive to discuss the matter in general as far as possible. To this end we find some sufficient conditions for (III.11a,b) which are probably almost necessary as well. This has the advantage of making (III.11b) more comprehensible. Let $w\,\mathrm{Re}B(w)=a+\phi(w)$, where the constant a may change when the sign of w changes. To begin with we assume that $\phi(w)$ vanishes at large $|w|$. As shown below, this is actually necessary for existence of solutions of the dispersion relation (II.10) if $\eta=O(\ln^{-\alpha}|w|)$, $\alpha>1$. Also, ϕ vanishes in at least one case in which η does not decrease at all; viz., the single-nucleon approximation ($\eta\equiv1$, $B=f^{(\mathrm{Born})}$). Thus, with $\phi\to0$, (III.11a) holds if and only if

$$\int_P dw[w^2\eta(w)]^{-1}<\infty\,. \qquad \text{(III.12)}$$

Let $f(w,w')$ be the integrand of (III.11b). Then it is necessary and sufficient for (III.11b) that the repeated integral exist:

$$\int_P dw\int_P dw'\,f(w,w')<\infty\,.$$

Therefore, (III.12) is necessary for (III.11b), just to ensure the convergence of the single integral over w'. However, it is not sufficient, as is seen by transforming to polar coordinates (r,θ). First change from w and w' to w and θ, and reverse the order of integrations. Then replace w by r. Both the legitimacy of reversing the order and the convergence of the double integral are assured if

$$\int_R^\infty rdr\,[\eta(r\cos\theta)\eta(r\sin\theta)]^{-1}\times\left[\frac{\phi(r\cos\theta)-\phi(r\sin\theta)}{r\cos\theta-r\sin\theta}\right]^2 \qquad \text{(III.13)}$$

converges uniformly for $0\leqslant\theta\leqslant2\pi$. It is sufficient to consider the first quadrant. We break it up into three regions of θ: $R_1=(0,\delta)$, $R_2=(\delta,\pi/2-\delta)$, $R_3=(\pi/2-\delta,\pi/2)$, $\delta<\pi/4$. Suppose $\phi(w)=O(w^{-\alpha}\ln^{-\beta}w)$, where α and β are chosen so that $w^{-\alpha}\ln^{-\beta}w$ is decreasing. Then in R_1 and R_3 the square bracket of (III.13), call it $\Phi(r\cos\theta,\,r\sin\theta)$, is $O(r^{-\alpha-1}\ln^{-\beta}r)$.

To handle R_2 notice that $\phi(w)$ has a continuous first derivative,[25] and, therefore, $\Phi(w,w')=\phi'(u)$, where u is some point between w and w'. But[26]

$$\phi'(w)=O(w^{-\alpha-1}\ln^{-\beta}w),$$

so

$$\Phi(r\cos\theta,\,r\sin\theta)=O(\rho^{-\alpha-1}\ln^{-\beta}\rho),$$

where ρ is the smaller of $r\cos\theta$, $r\sin\theta$. We use this bound in R_2, and obtain the result that

$$\Phi(r\cos\theta,\,r\sin\theta)=O(r^{-\alpha-1}\ln^{-\beta}r)$$

uniformly for $0\leqslant\theta\leqslant\pi/2$. Now if $1/\eta(w)=O(w^\gamma\ln^\delta w)$, (III.13) will converge uniformly if α, β, γ, δ are such that

$$\phi(w)/\eta(w)=O(\ln^{-\epsilon}w),\quad \epsilon>1/2.$$

To summarize, sufficient conditions for the L^2 properties (III.11a,b) are

(a) $\phi(w)=o(1)$,

(b) $\int_P dw[w^2\eta(w)]^{-1}<\infty$,

(c) $\phi(w)=O(|w|^{-\alpha}\ln^{-\beta}|w|)$,

(d) $\phi(w)/\eta(w)=O(\ln^{-\epsilon}|w|)$, $\epsilon>1/2$.

The exponents α, β, ϵ in (c) and (d) may be different in different quadrants of the w, w' plane, and α and β are to be such that $|w|^{-\alpha}\ln^{-\beta}|w|$ is decreasing. Condition (b) is also necessary for (III.11a,b).

We do not discuss in detail the complicated situation that arises if (a) is dropped. In that case, the L^2 property would depend on a cancellation between $\phi(r\cos\theta)$ and $\phi(r\sin\theta)$ in (III.13). That such a cancellation can occur is illustrated by the example $\phi(w)\sim\ln\ln w$. As we show presently [Eq. (III.14) ff] ϕ may behave as $\ln w$ at worst. With a behavior that strong it is difficult to see how the kernel could be L^2.

To derive restrictions on ϕ, as promised above, we return to the dispersion relation written in the form (III.11):

$$B(z)=f(z)-\frac{1}{\pi}\int_P dw\frac{\eta(w)\sin^2\delta(w)}{\kappa(w-z)}. \qquad \text{(III.14)}$$

From the unitarity bound $f(w)=O(|w|^{-1})$ and Appendix D, we have ϕ approaching zero and

$$\lim_{|w|\to\infty} w\,\mathrm{Re}B(w)=-\frac{1}{\pi}\int_P dw\frac{\eta(w)\sin^2\delta(w)}{\kappa}=a, \qquad \text{(III.15)}$$

[25] At least wherever $d\eta/dw\in H$, according to Appendix B. We assume that above some energy there are no points at which $d\eta/dw$ does not belong to H.

[26] From a generalized law of the mean one knows that if $f(x)$ and $g(x)$ have continuous first derivatives, then $f(x)/g(x)=f'(\theta x)/g'(\theta x)$, $0<\theta<1$, provided $f(0)=g(0)=0$ and the denominators do not vanish for $x\neq0$. If $|f(x)/g(x)|<M$ for all $x<x_0$, then $|f'(x)/g'(x)|<M$ for all $x<\theta_0x_0$, where $f(x_0)/g(x_0)=f'(\theta_0x_0)/g'(\theta_0x_0)$. Let $f(x)=\phi(x^{-1})$ and $g(x)=x^\alpha\ln^{-\beta}(x^{-1})$ to obtain the desired statement about $\phi'(x)$.

provided $\eta\sin^2\delta=O(\ln^{-\alpha}|w|)$, $\alpha>1$. Thus, if η itself is $O(\ln^{-\alpha}|w|)$, $\alpha>1$, ϕ vanishes. The strongest possible asymptotic behavior of ϕ occurs when $\eta\sin^2\delta$ approaches a nonzero constant. In that case, (III.14) and the theorem of Appendix A show that $\phi=O(\ln|w|)$.

To find a sort of converse to these remarks, we now look for implications of $w\,\mathrm{Re}B$ being bounded. The second term on the right of (III.14) is a Herglotz function (cf., Sec. VII). Therefore, an argument of Weinberg,[19] quoted in Sec. VII, shows that boundedness of $w\,\mathrm{Re}B$ implies the convergence of the integral of (III.15).

If $\int_P dw\eta(w)\kappa^{-1}$ does not converge (as in the elastic approximation $\eta\equiv 1$), then $\sin\delta(\infty)=\sin\delta(-\infty)=0$, just as in potential scattering. This is the case in the Born term model in which the contribution of a single Feynman graph (exclusive of isotopic spin factors) can be written as

$$w\,\mathrm{Re}B(w)=-\tfrac{1}{2}(g^2/4\pi)+O(\ln w^2/w). \quad \text{(III.16)}$$

Furthermore, there is basis for conjecture that δ approaching a multiple of π is a fairly general circumstance. In fact, this behavior is certainly present in situations other than the special case $\eta\equiv 1$, $w\,\mathrm{Re}B<\infty$, as we now prove. The easiest case to analyze is that in which η is asymptotic to a power of $\ln w$: $\eta(w)\sim c\ln^{-\alpha}w^2$. For simplicity we assume that c and α are the same at plus and minus infinity, but this restriction is easily discarded. Our first remark follows from the preceding observations: If $w\,\mathrm{Re}B$ is bounded and $\alpha\leqslant 1$, then $\sin\delta(\infty)=\sin\delta(-\infty)=0$. If $\alpha>1$ the same conclusion is reached if we add the following assumptions: (i) ϕ/η is bounded [but does not necessarily vanish as required in (d)]; (ii) $\delta(w)$ approaches a constant as $|w|\to\infty$. To prove this we apply the theorem of Appendix D to (III.14) and find

$$\phi(w)=w\eta\sin 2\delta/2\kappa+O(\ln^{-\alpha}w^2)+\chi(w),$$
$$\chi(w)=-\frac{1}{\pi}\left(\int_{-\infty}^{-w}+\int_{w}^{\infty}\right)\frac{dw\,\eta\sin^2\delta(w)}{\kappa}. \quad \text{(III.17)}$$

The point is that χ behaves as $\ln^{-\alpha+1}w^2$ if one does not have $\sin\delta(\infty)=\sin\delta(-\infty)=0$. But that would mean that ϕ/η is not bounded, contrary to hypothesis. To make explicit the asymptotic behavior of χ, we write it as

$$\frac{\chi(w)}{\psi(w)}\frac{\psi(w)}{\ln^{-\alpha+1}w^2}\ln^{-\alpha+1}w^2, \quad \text{(III.18)}$$

where

$$\psi(w)=\left(\int_{-\infty}^{-w}+\int_{w}^{\infty}\right)dw\,\eta\kappa^{-1}.$$

Evaluate the limit of the first factor of (III.18) by l'Hospital's rule. It is equal to $-\pi^{-1}[\sin^2\delta(-\infty)+\sin^2\delta(\infty)]$. Again by l'Hôpital, the limit of the second factor is a nonzero constant, and the proof is complete.

There is one further point which is important in the practical problem of finding suitable approximations for the interaction functions f^U and f^I. If $w\,\mathrm{Re}B$ is bounded and $\eta=O(\ln^{-\alpha}|w|)$, $\alpha>1$, the two terms wf^U and wf^I making up $w\,\mathrm{Re}B$ must be individually infinite at large w. This is apparent in (III.7). The term involving η in $wf^I(w)$ approaches a constant, by the theorem of Appendix D quoted above. By direct evaluation the other term is seen to have a logarithmic increase, which must be canceled by an opposite increase of wf^U. Evidently, the question of how to choose approximations for f^U and f^I is a delicate one.

The restriction (III.12) on the rate of decrease of η is something of a surprise. If (III.12) fails, the existence of a solution of the integral equation (III.8) is at least in doubt, although admittedly we cannot rule out solutions of a type not comprehended by the usual Fredholm theory.[27] Froissart,[28] who has given another solution of the problem of this section, found a similar restriction on η. He found it in a different way, however, and apparently regarded it more as a limitation of his method than as a hint that no solution may exist if η falls off too rapidly.

We may mention in passing that the method described here is somewhat easier to apply than that of Froissart. Besides, it seems to be a more systematic generalization of the elastic N/D procedure, and it throws into an interesting form the question of limitations on η.

In order to drop the restriction $f(z)=O(|z|^{-1})$, we consider the function

$$\Lambda(z)=N(z)-B(z)D(z)+\frac{1}{\pi}\int_P\frac{dw\,\mathrm{Im}D(w)\,\mathrm{Re}B(w)}{w-z}. \quad \text{(III.19)}$$

The integral converges, since $\mathrm{Im}D/w=O(|w|^{-\delta})$, $\delta>0$, and $w\,\mathrm{Re}B=O(\ln w^2)$ for any B consistent with the partial-wave dispersion relation. Since its discontinuity over the cuts vanishes, Λ is a polynomial. In fact, it is identically zero, since $\lim\Lambda(w)=0$, $w\to\infty$, by the theorem of Appendix D. Then if we let $z\to w+i0$ in (III.19) and take the real part we arrive immediately at (III.7). Note that the Poincaré-Bertrand formula is not necessary in this proof.

IV. THRESHOLD ZEROS AND EXISTENCE OF SOLUTIONS

In this section, we impose the physically reasonable requirement that the amplitude have the normal centrifugal barrier momentum dependence at thresholds. When this restriction is combined with the

[27] Some readers may be interested to know that Zaanen treats Fredholm equations in the general Lebesgue spaces L^P: A. C. Zaanen, *Linear Analysis* (Interscience Publishers, Inc., New York, 1953).

[28] M. Froissart, Nuovo Cimento **22**, 191 (1961).

unitarity condition, a strong, but different, restriction on the threshold behaviors of both f^U and B is implied. To see this, consider the function $f_{l+}(w)-f_{l+}{}^U(w)$ in a state of definite isotopic spin. It has no unphysical singularities. Equation (II.10) serves to express it as an integral over P with a non-negative spectral function $\mathrm{Im}f_{l+}(w)$. It follows that $f_{l+}-f_{l+}{}^U$ is monotonically increasing in the interval $-w_0\leqslant w\leqslant w_0$ and hence vanishes at most once in this interval. Now for $l\geqslant 1$, $f_{l+}(w)$ vanishes at $w=\pm w_0$ and $w=\pm(M-m)$; cf. Appendix E. Consequently, $f_{l+}{}^U(w)$ can vanish at no more than one of these points. For $l=0$, $f_{0+}{}^U(w)$ can vanish at no more than one of the two points $w=-w_0$, $w=-(M-m)$ at which $f_{0+}(w)$ vanishes. Equation (II.11) can be used to show that the same statements also hold for $B(w)$.

This necessary condition is not fulfilled if, for example, the Born approximation is used for $f_{l+}{}^U(w)$. It is clear, therefore, that no ghost-free solution exists for this model. The conclusion holds for arbitrary, nonvanishing coupling strength and for arbitrary inelastic effects. Our remarks don't apply directly to the model considered by Uretsky[10] because he treated only the S wave, mutilating the $P_{1/2}$ scattering in a way that is inconsistent with our point of view.

The restriction on threshold behavior can be contrasted to the limitations on asymptotic behavior that also follow from unitarity. A violation of the latter is often traced to the fact that the discontinuity function Δf increases too rapidly at infinity. On the other hand, the restriction on threshold behavior involves only a finite number of zeros. Any model that violates only this restriction can be "corrected" simply by adding to $f_{l+}{}^U(w)$ a function with only a finite number of poles. The question of whether these additional poles are authentic contributions to $f_{l+}{}^U(w)$ must be examined in individual cases. In the limited class of models based on the Cini-Fubini representation,[11] a general answer will now be given.

In the construction of models, we need the energy-square variables $\bar{s}$ and t in the two crossed channels. They are related to s by $s+\bar{s}+t=2M^2+2m^2$ and $t=-2k^2(1-\cos\theta)$, where θ is the scattering angle in the s channel. Consider the invariant amplitude $A^{(\pm)}(s,t)$. The Cini-Fubini approximation to the Mandelstam representation is (for pion-nucleon scattering)

$$\begin{aligned}A^{(\pm)}(s,t) &= \frac{1}{\pi}\int_{s_0}^{s_m} ds'\,\mathcal{A}_s{}^{(\pm)}(s',t)\left[\frac{1}{s'-s}\pm\frac{1}{s'-\bar{s}}\right] \\ &\quad +\frac{1}{\pi}\int_{4m^2}^{t_m} dt'\,\mathcal{A}_t{}^{(\pm)}(t',s-\bar{s})\frac{1}{t'-t} \\ &\quad +\text{a low-order polynomial in } s,\ \bar{s},\text{ and } t. \end{aligned}\tag{IV.1}$$

Here $\mathcal{A}_s{}^{(\pm)}$ and $\mathcal{A}_t{}^{(\pm)}$ are roughly the absorptive parts for the s and t channels, respectively. The approximation consists in treating the dependence on the second argument of the absorptive parts as a low-order polynomial, corresponding to the first few partial waves in each channel. Let $[f_{l+}{}^U(w)]_{\bar{s}+t}$ be the s-channel partial-wave projection (taking spin and kinematic factors into account) of the terms in (IV.1) that have $s'-\bar{s}$ and $t'-t$ denominators. It is well known, and easily verified by expanding about $k^2=0$ and using the orthogonality of Legendre polynomials, that such functions have zeros at thresholds. Therefore, if $[f_{l+}{}^U(w)]_{\bar{s}+t}$ alone is used as the model for the unphysical singularities, the necessary threshold property is violated and there are no ghost-free solutions.

It is easy to construct models which avoid this difficulty. Let $[f_{l+}{}^U(w)]_s$ be the contribution from the terms in (IV.1) with $s'-s$ denominators. The essential point is seen more clearly if we neglect spin complications; then we have

$$\begin{aligned}[f_{l+}{}^U(w)]_s &= \frac{1}{\pi}\int_{w_0}^{\infty} dw'(w'-w)^{-1}\sum_{l'}(2l'+1)\,\mathrm{Im}f_{l'}(w') \\ &\quad\times\int_{-1}^{1} dx\,P_l(x)[P_{l'}(1-(k^2/k'^2)(1-x))-P_{l'}(x)],\end{aligned}\tag{IV.2}$$

where $P_l(x)$ is a Legendre polynomial. For a finite sum on l', the right-hand side is analytic as a function of w, except for poles at $w=0$, arising from the k-dependent $P_{l'}$. Thus, $[f_{l+}{}^U(w)]_s$ is a rational function. If spin effects are included, the contribution to $[f_{0+}{}^U(w)]_s$ from the (3,3) resonance behaves as w^{-4} near $w=0$ and doesn't vanish at any threshold. This term is present in the static model, where it participates in a striking cancellation. In general, there is no reason for $[f_{l+}{}^U(w)]_s$ to vanish at any threshold. Thus models with consistent threshold behavior are comprehended in the Cini-Fubini framework. The singularity at $w=0$, however, depends on all waves of the s channel (that are included in the l' sum), so the calculation of a given partial wave is no longer decoupled from the remaining ones.

The interdependence of the different partial waves and the more quantitative restrictions on f^U and B will be brought out by the following discussion.

The point is that for a given $[f^U]_{\bar{s}+t}$ and η, the threshold requirement implies some relations among the parameters γ_n of $[f_{l+}{}^U]_s=\sum\gamma_n w^{-n-1}$, $n=0,1,\cdots$. With some courage, the method we propose can be applied to more general cases, but to keep the motivation clear, we develop it in the Cini-Fubini framework.

Threshold functions $\theta_l(w)$ are defined by

$$\theta_0(w)=w^2(w+M+m)^{-1}(w+M-m)^{-1}\tag{IV.3a}$$

for $l=0$, and for $l\geqslant 1$, by

$$\begin{aligned}\theta_l(w) &= w^{2l+3}(w^2-(M-m)^2)^{-1} \\ &\quad\times(w-w_0)^{-l}(w+w_0)^{l+1}.\end{aligned}\tag{IV.3b}$$

Note that $\theta_l(w)=O(1)$ for large w. The centrifugal barrier threshold behavior of $f_{l+}(w)$ is enforced by applying the N/D method to the function

$$h_l(w)=\theta_l(w)f_{l+}(w)=[\theta_l(w)N(w)]D^{-1}(w)$$

and seeking a solution $\theta_l(w)N(w)$ that is finite at threshold. The integral equation for

$$n(w)=2\,\mathrm{Re}N(w)[1+\eta(w)]^{-1}$$

is (suppressing l)

$$\eta(w)n(w)=\mathrm{Re}\tilde{B}(w)+\frac{1}{\pi}\int_P dw'\frac{\kappa' n(w')}{w'}\times\left[\frac{w'\theta(w')\theta^{-1}(w)\,\mathrm{Re}\tilde{B}(w')-w\,\mathrm{Re}\tilde{B}(w)}{w'-w}\right], \quad \text{(IV.4)}$$

where $\tilde{B}$ is a modified interaction function defined by

$$\theta(w)\tilde{B}(w)=\frac{1}{\pi}\int_U dz\frac{\Delta f(z)\theta(z)}{z-w}+\frac{1}{2\pi}\int_P dw'\frac{[1-\eta(w')]\theta(w')}{\kappa'(w'-w)}. \quad \text{(IV.5)}$$

Now B and $\tilde{B}$ differ by an additive rational function which need not vanish at any threshold. In fact, $\tilde{B}$ has the centrifugal barrier zeros and, therefore, n or N does.[28a] However, unless some accidental cancellation occurs, the resulting solution $f=h\theta^{-1}$ will have poles at $w=0$ in addition to any that might already be present in f^U. The new poles (similar to ghosts) amount to a modification of the coefficients γ_n and are not objectionable to the extent that the γ_n are considered ambiguous.

The expression for $\tilde{B}$ can be simplified considerably if we write $\Delta f=[\Delta f]_{\bar{s}+t}+[\Delta f]_s$. Since $[f]_{\bar{s}+t}$ has the centrifugal barrier behavior already, the Cauchy relation for $\theta(w)[f(w)]_{\bar{s}+t}$ shows that

$$\pi^{-1}\int_U dz\,\theta(z)(z-w)^{-1}[\Delta f(z)]_{\bar{s}+t}=\theta(w)[f(w)]_{\bar{s}+t}. \quad \text{(IV.6)}$$

The contribution from $[\Delta f]_s$, where

$$[f(w)]_s=\sum\gamma_n w^{-n-1},$$

can be written as

$$\frac{1}{\pi}\int_U dz\frac{\theta(z)[\Delta f(z)]_s}{z-w}=-\sum_{n=0}^{\infty}\frac{\gamma_n}{n!}\frac{d^n}{dz^n}\left(\frac{\theta(z)}{z-w}\right)\Bigg|_{z=0}=\sum_{m=0}^{\infty}\Gamma_m w^{-m-1}, \quad \text{(IV.7)}$$

where the Γ_m are defined here in terms of the γ_n. Due to θ, the Γ_m depend on γ_n only for $n>2l+3-\delta_{l0}$. That is, the γ_n for $n\leqslant 2l+3-\delta_{l0}$ do not enter into the integral equation. The low-order poles of $[f]_s$ are constructed by the N/D method through the zeros of θ and, in effect, the coefficients γ_n for $n\leqslant 2l+3-\delta_{l0}$ are automatically assigned values that yield the proper threshold behavior. For example, take $J=\frac{1}{2}(l=0)$ and assume $\gamma_n\neq 0$ for $n=0$, 1, 2, and 3. This includes contributions from the (3,3) state and second πN resonance in the s channel. Then

$$\begin{aligned}\Gamma_0&=\gamma_2(M^2-m^2)^{-1}+2M\gamma_3(M^2-m^2)^{-2},\\ \Gamma_1&=-\gamma_3(M^2-m^2)^{-1},\end{aligned} \quad \text{(IV.8)}$$

and $\Gamma_m=0$ for $m\geqslant 2$. The calculation of a given partial-wave is independent of the calculation of the remaining ones in the same channel to the extent that the solution may not be sensitive to higher values of n.

V. A UNIQUENESS THEOREM

We choose to admit only scattering amplitudes that have the proper threshold zeros, that are uniformly bounded by a polynomial as w approaches infinity in any complex direction, and which have phase shifts with real parts approaching finite constants for large w. Suppose $f_{l+}^{(1)}$ and $f_{l+}^{(2)}$ are two such functions that satisfy the partial-wave dispersion relation with the same given f_{l+}^U, η_{l+}, and $\eta_{(l+1)-}$. We seek conditions under which the difference $g(w)=f_{l+}^{(1)}(w)-f_{l+}^{(2)}(w)$ vanishes identically.

Now $g(z)$ is analytic except for the two cuts comprising P and perhaps poles if $f_{l+}^{(1)}$ and $f_{l+}^{(2)}$ have distinct bound states.

The boundary values of $g(z)$ are given by

$$\lim_{z\to w+i0}g(z)=\eta_{l+}(w)k^{-1}(w)\times\exp i\,\mathrm{Re}(\delta_{l+}^{(1)}+\delta_{l+}^{(2)})\times\sin\mathrm{Re}(\delta_{l+}^{(1)}-\delta_{l+}^{(2)}), \quad \text{(V.1a)}$$

and

$$\lim_{z\to -w-i0}g(z)=-\eta_{(l+1)-}(w)k^{-1}(w)\times\exp i\,\mathrm{Re}(\delta_{(l+1)-}^{(1)}+\delta_{(l+1)-}^{(2)})\times\sin\mathrm{Re}(\delta_{(l+1)-}^{(1)}-\delta_{(l+1)-}^{(2)}), \quad \text{(V.1b)}$$

where $w>w_0$ and $\delta^{(i)}=\delta^{(i)}(+w)$ is the phase shift of $f^{(i)}$, $i=1$, 2. Using $\mathfrak{D}$ we construct $\Phi(z)$ as

$$\Phi(z)=g(z)\mathfrak{D}(z;\mathrm{Re}(\delta_{l+}^{(1)}+\delta_{l+}^{(2)}),\mathrm{Re}(\delta_{(l+1)-}^{(1)}+\delta_{(l+1)-}^{(2)}))\times\prod_B(z-w_B^{(1)})(z-w_B^{(2)}). \quad \text{(V.2)}$$

[28a] Even though $1/\theta_1$ behaves as $\omega^2-(M-m)^2$ at $\omega=\pm(M-m)$, the amplitude $f_{1+}(\omega)$ constructed by the method described will have the correct behavior $[\omega^2-(M-m)^2]^{1/2}$ found in Appendix E, provided only that the given discontinuity Δf is proportional to $[\omega^2-(M-m)]^{1/2}$ near $\pm(M-m)$. This is because $\theta(\omega)N(\omega)$ involves the term $\int dz\Delta f(z)\theta(z)D(z)/(z-\omega)$ which has a singularity of the type $[\omega^2-(M-m)^2]^{-1/2}$.

For notation see (II.14). The coincidence of $W_B^{(i)}$ with any bound-state poles of $f^{(i)}$ and the phase conditions on g and $\mathfrak{D}$ ensure that Φ has no singularities in any finite region. For large z, $\mathfrak{D}$ is uniformly bounded by a polynomial (Appendix A). Since we assume $f^{(1)}$ and $f^{(2)}$ are uniformly bounded by a polynomial for large z, $g(z)$ is also. Hence, $|\Phi(z)|=O(|z|^m)$ for some integer m, and by a familiar theorem $\Phi(z)$ is a polynomial. Let φ be the degree of Φ. Because of the Hölder condition on $\mathrm{Re}\delta$, the exponent in the definition of $\mathfrak{D}$ is finite for all finite z, including real values on P. Hence $\mathfrak{D}$ and $\mathfrak{D}^{-1}$ have no finite zeros and the zeros of Φ are identical with those of g. The number of these zeros is φ. The remainder of the proof consists of finding upper and lower bounds on φ.

The upper bound depends on the limiting values $\mathrm{Re}\delta_{l\pm}(\infty)=\lim_{w\to\infty}\mathrm{Re}\delta_{l\pm}(w)$. Then for large, real w, $|\mathfrak{D}(w+i0)|=O(w^{q+\epsilon})$, where

$$q=\pi^{-1}\,\mathrm{Re}[\delta_{l+}{}^{(1)}(\infty)+\delta_{l+}{}^{(2)}(\infty)+\delta_{(l+1)-}{}^{(1)}(\infty)+\delta_{(l+1)-}{}^{(2)}(\infty)],$$

and ϵ is any real number greater than zero. If $f^{(i)}$ has $n^{(i)}$ bound-state poles, all distinct, the polynomial factor in (V.2) gives a power behavior of degree $n^{(1)}+n^{(2)}$ for large w. Finally, the unitarity condition gives $|g(w+i0)|=O(w^{-1})$ and we have

$$\varphi\leqslant q+n^{(1)}+n^{(2)}-1. \tag{V.4}$$

For if $\varphi>q+n^{(1)}+n^{(2)}-1$, there is a contradiction of the statement $|\Phi(z)|=O(|z|^{q+n(1)+n(2)-1+\epsilon})$, because ϵ can be taken so small that

$$q+n^{(1)}+n^{(2)}-1+\epsilon<\varphi.$$

To find a lower bound on φ, we evaluate the number of known zeros of $g(z)$. The centrifugal barrier behavior of both $f^{(1)}$ and $f^{(2)}$ implies that $g(z)$ has an lth order zero at $w=w_0$, and an $(l+1)$th order zero at $w=-w_0$. Such a behavior follows from the Mandelstam representation (see Appendix E) and is demanded in any theory in which the forces are limited in range. In addition, crossing symmetry and unitarity imply zeros at the other roots $w=\pm(M-m)$ of $k^2(w)=0$. In fact, $g(z)$ has a zero at $w=-(M-m)$ for all l, and a zero at $M-m$ for $l\neq 0$, as shown in Appendix E. From (V.1) $g(z)$ has a zero on P at each point where $\mathrm{Re}(\delta_{l+}{}^{(1)}-\delta_{l+}{}^{(2)})$ or $\mathrm{Re}(\delta_{(l+1)-}{}^{(1)}-\delta_{(l+1)-}{}^{(2)})$ passes through an integral multiple of π. On the right-hand physical cut there are at least $\pi^{-1}\{|\mathrm{Re}[\delta_{l+}{}^{(1)}(\infty)-\delta_{l+}{}^{(2)}(\infty)]|\}$ zeros of this kind, where $\{C\}$ is "the greatest integer less than C."

At this point, we first complete the proof under the restrictive assumption that there are no bound states, or, more generally, that the bound-state poles of $f^{(1)}$ and $f^{(2)}$ coincide in position and residue. Then

$$\varphi\geqslant 2l+1+(2-\delta_{l0})+\pi^{-1}\{|\mathrm{Re}[\delta_{l+}{}^{(1)}(\infty)-\delta_{l+}{}^{(2)}(\infty)]|\}+\pi^{-1}\{|\mathrm{Re}[\delta_{(l+1)-}{}^{(1)}(\infty)-\delta_{(l+1)-}{}^{(2)}(\infty)]|\}. \tag{V.5}$$

The inequalities Eqs. (V.4) and (V.5) can be solved to find a relation for $\delta^{(1)}$ and $\delta^{(2)}$ separately. If we set

$$\mathrm{Re}[\delta_{l+}{}^{(1)}(\infty)-\delta_{l+}{}^{(2)}(\infty)]=x,$$

and

$$\mathrm{Re}[\delta_{(l+1)-}{}^{(1)}(\infty)-\delta_{(l+1)-}{}^{(2)}(\infty)]=y,$$

the algebra can be done in the form $\omega\geqslant|x|+|y|-x-y\geqslant 0$. The inequality $\omega\geqslant 0$ reads

$$p^{(i)}=\pi^{-1}\,\mathrm{Re}[\delta_{l+}{}^{(i)}(\infty)+\delta_{(l+1)-}{}^{(i)}(\infty)]\geqslant l+1-\tfrac{1}{2}\delta_{l0}. \tag{V.6}$$

If there are at least two admissible solutions of the partial-wave dispersion relation which are free of bound states, then (V.6) is a necessary condition on any such solution. Since $g(z)\equiv 0$ is the only alternative to (V.6), we have a kind of uniqueness theorem: A solution that contradicts (V.6) is the only admissible solution without a bound state.

Is there ever an admissible solution without bound states that contradicts (V.6)? For $l\neq 0$ the answer is likely to be yes. In fact, if η does not decrease too rapidly, an admissible solution constructed from an L^2 solution of the N/D equation (III.9) will have the desired property, provided it has no bound state. Suppose that $\eta(w)|w|^\alpha\to\infty$ as $|w|\to\infty$, where $0<\alpha<1$. Since $x\in L^2$ means that $\int_P dw\,\eta(w)n^2(w)<\infty$, it follows that $n^2(w)|w|^{1-\alpha}=o(1)$. Since $\mathrm{Im}D=-\kappa n$, the Cauchy representation of D and the work of Appendix D show that $D=O(|w|^{(1+\alpha)/2})$. Since $D=\Phi\mathfrak{D}$, where Φ is a polynomial, we can conclude from Appendix A, Eq. (A7), that $p<1$ and (V.6) is contradicted. If $l=0$ the same argument does not quite suffice. However, if one finds $n(w)=O(|w|^{-\alpha})$, $\alpha>\frac{1}{2}$, then $p<\frac{1}{2}$ and the corresponding solution is a unique $l=0$ solution in the class considered.

If there are two solutions $f^{(1)}$, $f^{(2)}$, and each has $n=n^{(1)}=n^{(2)}$ bound states, then (V.6) is replaced by $p^{(i)}\geqslant l+1-\frac{1}{2}\delta_{l0}-n$. If the latter inequality is violated by some solution, it is the only solution with n bound states. A better theorem for the situation in which bound states are allowed seems possible only if we can claim additional zeros of $g(z)$. One basis for doing this is provided by the assumption that the residue of a bound-state pole has a definite sign; i.e., $f^{(1)}\approx R_B(w_B{}^{(1)}-w)^{-1}$ near $w=w_B{}^{(1)}$, where $R_B>0$. Then $f^{(1)}$ must have a zero between consecutive poles. The number of such zeros in $g(z)$ depends on $\mu=|n^{(1)}-n^{(2)}|$ because $g(z)$ need not vanish between adjacent poles of $f^{(1)}$ and $f^{(2)}$. The counting is complicated by the fact that such zeros might coincide with the ones at $w=\pm(M-m)$ already counted. In any case, there are at least $\mu-3+2\delta_{\mu 0}+\delta_{\mu 1}+\delta_{l0}$ additional zeros. Altogether, the minimum estimate for φ is

$$\varphi \geqslant 2l+1+(2-\delta_{l0}) +\pi^{-1}\{|\mathrm{Re}[\delta_{l+}^{(1)}(\infty)-\delta_{l+}^{(2)}(\infty)]|\} +\pi^{-1}\{|\mathrm{Re}[\delta_{(l+1)-}^{(1)}(\infty)-\delta_{(l+1)-}^{(2)}(\infty)]|\} +|n^{(1)}-n^{(2)}|+2\delta_{\mu 0}+\delta_{\mu 1}+\delta_{l0}-3. \quad \text{(V.7)}$$

The inequalities (V.5) and (V.7) can be put in the form $\omega' \geqslant |x|+|y|+\mu-x-y-(n^{(1)}-n^{(2)})\geqslant 0$. Thus, for both $i=1$, 2 we have

$$\pi^{-1}\,\mathrm{Re}[\delta_{l+}^{(i)}(\infty)+\delta_{(l+1)-}^{(i)}(\infty)]+n^{(i)} \geqslant l-\tfrac{1}{2}+\delta_{\mu 0}+\tfrac{1}{2}\delta_{\mu 1} \geqslant l-\tfrac{1}{2}. \quad \text{(V.8)}$$

The connection with potential scattering is interesting. In this case, our inequality becomes $\delta_l(\infty) \geqslant \pi(\frac{1}{2}l-1-n_B)$ where n_B is the number of bound states. On the other hand, Levinson's theorem[29] states that $\delta_l(\infty)=-\pi n_B$. Thus, there is a contradiction for $l\geqslant 3$. Either there is a unique, admissible solution of the partial-wave dispersion relations of potential theory for $l\geqslant 3$, or else our assumption about the sign of bound-state residues is not acceptable.

There is another slight extension. If additional parameters are granted, in the form that f_{l+} (both $f_{l+}^{(1)}$ and $f_{l+}^{(2)}$) has a prescribed value at h different points, then $g(z)$ has h additional zeros at these points, and an added term $\frac{1}{2}h$ is implied on the righthand sides of Eqs. (V.6) and (V.8).

VI. CASTILLEJO-DALITZ-DYSON AMBIGUITY

To include the CDD ambiguity in the N equation we generalize the method described at the end of Sec. III. For this purpose we define functions N and D with the same restrictions on asymptotic behavior as the N and D of that section. Of course, the new functions have poles. Amplitudes in the class for which $n\pi \leqslant p<(n+1)\pi$ have a decomposition N_0/D_0 in which D_0 has no poles and is $O(|z|^{n+1-\delta})$, $\delta>0$. D_0 differs from $\mathfrak{D}$ of (II.13) by at most a multiplicative constant. Now $D=D_0/\rho$, where $\rho=\prod_{i=1}^n(z-z_i)$, is $O(|z|^{1-\delta})$. If we introduce the corresponding numerator function $N=D_0/\rho$, the function $\Lambda(z)$ of (III.19) can be constructed. Λ is well defined if the z_i do not lie on P. But for amplitudes in the class $n\pi<p<(n+1)\pi$ there are n points w_i at which $\sin\delta(w_i) =\mathrm{Im}D_0(w_i)/|D_0(w_i)|=0$; (we set aside for the moment the case $p=n\pi$). It is convenient, and in accord with the work of Castillejo *et al.*, to let the z_i approach w_i. Then since $\mathrm{Im}D_0(w_i)=0$, the limit $z_i\rightarrow w_i$ causes no trouble in the dispersion relation for D. In fact, we know from Appendix B, Lemma A, that $\mathrm{Im}D_0/(w-w_i)$ satisfies a Hölder condition near w_i provided that $d\delta/dw$ does. We assume that indeed $d\delta/dw\in H$ near w_i. The assumption is certainly reasonable if w_i is not an S-wave, two-body threshold. Thus, $\mathrm{Im}D\in H$ on P, which means that the Cauchy representation of D and the function Λ are both well-defined for $z_i=w_i$. Now, Λ has zero jump over the cuts U and P, so its only singularities are the poles of D which make their appearance in the $-BD$ term of (III.19). D now takes the form

$$D(z)=1+z\left[\sum_{i=1}^{n}\frac{c_i}{z-w_i}-\frac{1}{\pi}\int_P dw\,\frac{\kappa n(w)}{w(w-z)}\right], \quad \text{(VI.1)}$$

where $-\kappa n(w)=\mathrm{Im}D(w)$ and

$$c_i=c_i^*=D_0(w_i)[w_i\prod_{j\neq i}(w_i-w_j)]^{-1},$$

$$D_0(0)[\prod_j(-w_j)]^{-1}=1.$$

If $n=1$, then $c_1=D_0(w_1)/w_1$. When (VI.1) is substituted in (III.19) one sees that

$$\Lambda(z)=-\sum_{i=1}^{n}\frac{c_iw_i\,\mathrm{Re}B(w_i)}{z-w_i}. \quad \text{(VI.2)}$$

In (VI.2) let $z\rightarrow w\in P$ and take the real part. The result is a simple modification of (III.7):

$$\eta(w)n(w)=\mathrm{Re}B(w)+\sum_{i=1}^{n}c_i\frac{w\,\mathrm{Re}B(w)-w_i\,\mathrm{Re}B(w_i)}{w-w_i} +\frac{1}{\pi}\int_P dw'\,\frac{\kappa'}{w'}n(w')\left[\frac{w'\,\mathrm{Re}B(w')-w\,\mathrm{Re}B(w)}{w'-w}\right]. \quad \text{(VI.3)}$$

The $2n$ real parameters c_i, w_i, are to be regarded as completely arbitrary unless they are associated with unstable elementary particles with known masses and widths.[1,7,12] According to (III.1) and (III.2), the condition $\mathrm{Im}D_0(w_i)=0$ means that $f(w_i)=i(1-\eta)(2\kappa)^{-1}$. The CDD points correspond to zeros of the amplitude only below the inelastic threshold.

If $p=n\pi$ there may be only $n-1$ finite points w_i at which $\sin\delta(w_i)=0$. In that event (VI.3) does not necessarily hold, and we must resort to the more subtle analysis of Sec. VII. According to the remarks preceding (VII.6), any amplitude[30] of the class $n\pi\leqslant p<(n+1)\pi$ has an N/D decomposition in which[31]

$$D(z)=1+z\left[A+\sum_{i=1}^{n}\frac{c_i}{z-w_i}-\frac{1}{\pi}\int_P dw\,\frac{\kappa}{w}\frac{n(w)}{w-z}\right], \quad \text{(VI.4)}$$

where $A=A^*$, and either A or one of the c_i is zero. The zA term may be regarded as a CDD pole at infinity. It may be nonzero only if $p=n\pi$, and it need

[29] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **25**, No. 9 (1949); R. Haag, Nuovo Cimento **5**, 203 (1957).

[30] That is, any amplitude for which the phase $\delta(w)$ does not go through a multiple of π an infinite number of times.

[31] We shall suppose that $n(w)\in H$. This does not follow from the work of Sec VII. It presumably is implied by $\delta(w)\in H$, but we shall not try to prove that here.

not be present in that case if $\pi^{-1}[\delta_{l+}(w)+\delta_{(l+1)-}(w)]$ approaches $p=n\pi$ from above. Now when (VI.4) is substituted in (III.19) we must somehow guarantee the convergence of the integral in (III.19). An integration by parts shows that one has convergence if

$$w\,\mathrm{Re}B(w)=O(1), \qquad \text{(VI.5a)}$$

and

$$\int_P |[w\,\mathrm{Re}B(w)]'|\,dw=<\infty. \qquad \text{(VI.5b)}$$

Condition (VI.5a) is normally expected, as is indicated by the discussion of Sec. III. Equation (VI.5b) follows from (VI.5a) and the condition $\eta(w)=O(\ln^{-\alpha}|w|),\alpha>1$, according to (III.17). In any event, let us suppose that the integral of (III.19) converges, either through satisfaction of (VI.5) or otherwise. Then $\Lambda(z)$ is a rational function as before. $\Lambda(w)$ is $o(w^\alpha)$, $0<\alpha<1$, and since it is rational it must be bounded for all $|w|$ greater than some W. Its value at infinity is

$$\lim_{w\to\infty}\left\{-w\,\mathrm{Re}B(w)A+\left[\eta(w)n(w)-\frac{1}{\pi}\int_P dw'\,\frac{\kappa'}{w'}\times n(w')\left(\frac{w'\,\mathrm{Re}B(w')-w\,\mathrm{Re}B(w)}{w'-w}\right)\right]\right\}<\infty. \qquad \text{(VI.6)}$$

If the limits of the two terms of (VI.6) exist separately, our integral equation for n may be written

$$\begin{aligned}\eta(w)n(w)&\\ =\mathrm{Re}B(w)&+A[w\,\mathrm{Re}B(w)-\lim_{w\to\infty}w\,\mathrm{Re}B(w)]\\ &+\sum_{i=0}^{n}c_i\frac{w\,\mathrm{Re}B(w)-w_i\,\mathrm{Re}B(w_i)}{w-w_i}\\ &+\frac{1}{\pi}\int_P dw'\,\frac{\kappa'}{w'}n(w')\left[\frac{w'\,\mathrm{Re}B(w')-w\,\mathrm{Re}B(w)}{w'-w}\right]\\ &\times\lim_{w\to\infty}\left\{\eta(w)n(w)-\frac{1}{\pi}\int_P dw'\,\frac{\kappa'}{w'}n(w')\right.\\ &\qquad\left.\times\left[\frac{w'\,\mathrm{Re}B(w')-w\,\mathrm{Re}B(w)}{w'-w}\right]\right\}.\end{aligned} \qquad \text{(VI.7)}$$

The last term on the right of (VI.7) may or may not be zero, depending on the rate of decrease of $n(w)$ at infinity. But according to the construction of D in terms of $\mathfrak{D}$ in Sec. VII, the large w behavior of $n(w)$ depends on the rate at which $\pi^{-1}[\delta_{l+}(w)+\delta_{(l+1)-}(w)]$ approaches $n\pi$. If the approach is rapid, the last term in (VI.7 is zero and only $2n-1$ CDD parameters enter. If the approach is slow, the full complement of $2n$ CDD parameters may enter (VI.7), even only $2n-1$ appeared in (VI.4). Thus, for a restricted class of amplitudes the CDD pole at infinity implies only one arbitrary constant, but in general two are involved. Note that equation (VI.7) is of Fredholm type only if the last term on the right side is zero and the factor multiplying A decreases rapidly enough at large w.

VII. THE HERGLOTZ DENOMINATOR

It is interesting to observe that the partial-wave scattering amplitude can always be factored in the form N/D in such a way that D is a so-called Herglotz function. A function $H(z)$ analytic in the upper half plane is called a Herglotz function[32] if it has the property

$$\mathrm{Im}H(z)\geqslant 0,\quad \mathrm{Im}z>0.$$

The Wigner R function $R(z)$ is a special case in which $R(z)$ is also meromorphic in the complete z plane.[33] Herglotz functions, or generalized R functions, have been studied in the theory of moments,[34] the theory of electrical circuits,[35] the analysis of Low's scattering equation,[2] field theory,[36] and in the proof of the Pomeranchuk theorem.[19] We collect here the principal properties of Herglotz functions.

Theorem. If $H(z)$ is analytic in the half-plane $\mathrm{Im}z>0$, and if $\mathrm{Im}H(z)\geqslant 0$ for $\mathrm{Im}z>0$, then there exists a bounded nondecreasing real function $\alpha(w)$ such that

$$H(z)=Az+c+\int_{-\infty}^{\infty}d\alpha(w)(1+wz)(w-z)^{-1}, \qquad \text{(VII.1)}$$

where A and c are real and $A\geqslant 0$. Moreover,

$$\lim_{z\to\infty}z^{-1}H(z)=A,$$

when $z\to\infty$ along any direction not parallel to the real axis.

It follows that $H(z^*)=H^*(z)$, $\mathrm{Im}H(z)>0$ for $\mathrm{Im}z>0$ and $H(z)$ has no complex zeros. Note that $-H^{-1}(z)$ is also Herglotz. If $\mathrm{Im}H(w)$ vanishes for real w in some interval $-\mu<w<\mu$, then $\int w^{-1}d\alpha(w)$ exists and Eq. (VII.1) can be put in the form[37]

$$H(z)=H(0)+Az+z\int_{-\infty}^{\infty}d\beta(w)\,w^{-1}(w-z)^{-1}, \qquad \text{(VII.1a)}$$

where $d\beta(w)=(1+w^2)d\alpha(w)$. Weinberg[19] has shown that at least one of the integrals

$$\int_{-\infty}^{\infty}w^{-1}d\beta(w),\quad \int_{-\infty}^{\infty}w^{-1}|H(w+i0)|^{-2}d\beta(w),$$

[32] A. Herglotz, Ber. Verhandl. Sachs. Akad. Wiss. Leipzig, Math Naturw. Kl. **63** (1911).
[33] E. P. Wigner, Ann. Math. **53**, 36 (1951).
[34] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943); cf., especially p. 23.
[35] P. I. Richards, Quart. J. Appl. Math. **6**, 21 (1948); T. T. Wu, J. Math. Phys. **3**, 262 (1962).
[36] K. Symanzik, J. Math. Phys. **1**, 249 (1960), Appendix B.
[37] Cf., reference 2, Eq. (2.13).

TABLE I. Topological types.

Type	$\hat{\delta}_{l+}$	$\hat{\delta}_{(l+1)-}$	ϵ_0	σ	$(2J,2I)\pi N$ state (empirical)
$\tau(+-)$	>0	<0	0	-1	(1,1) N
$\tau(-+)$	<0	>0	0	$+1$	(3,1) N^{**}
$\tau(++)$	>0	>0	-1	-1	(3,3) N^*
$\tau(--)$	<0	<0	$+1$	$+1$	(1,3)

must converge. Symanzik[36] and Weinberg[19] have shown that if $D_1(z)=D_1^*(z^*)$ is uniformly bounded by a polynomial and is analytic except for singularities on the real axis and if its spectral function $\text{Im}D_1(w+i0)$ has at most a finite number of zeros, then $D_1(z)=R(z)H(z)$, where $R(z)$ is a rational function and $H(z)$ is Herglotz.

To demonstrate the existence of a Herglotz denominator we examine all points on P at which $\text{Re}\delta$ passes through an integral multiple of π. Corresponding to the continuous phase $\text{Re}\delta_{l\pm}$, let us define $\hat{\delta}_{l\pm}$ by[38]

$$\hat{\delta}_{l\pm}(w)=\text{Re}\delta_{l\pm}(w)-\pi\sum_j\Theta(w-\bar{w}_j)+\pi\sum_i\Theta(w-\hat{w}_i),\quad\text{(VII.2)}$$

where $\bar{w}_j$, $j=1,2,\cdots,j(\max)$, [resp. $\hat{w}_i$, $i=1,\cdots,i(\max)$] are the energies at which $\text{Re}\delta_{l\pm}$ goes up (resp. down) through an integral multiple of π, and $\Theta(w)=\frac{1}{2}(1+|w|w^{-1})$. We assume $i(\max)$ and $j(\max)$ are finite. It is easy to see that

$$0\leqslant\hat{\delta}_{l\pm}(w)\leqslant\pi,\quad\text{(VII.3a)}$$

if $\hat{\delta}_{l\pm}$ is positive just above threshold, or

$$-\pi\leqslant\hat{\delta}_{l\pm}(w)\leqslant 0\quad\text{(VII.3b)}$$

if it is negative there. A systematic characterization can be made if we distinguish four cases according to the signs of $\hat{\delta}_{l+}$ and $\hat{\delta}_{(l+1)-}$. We call these "topological types" and denote them by $\tau(++)$, etc., as shown in Table I. We don't know that there is any significance to this classification, but it is interesting to note that, empirically, each type occurs once in $J=1/2, 3/2$; $I=1/2, 3/2$ pion-nucleon scattering.

We now define the Herglotz denominator explicitly in terms of $\mathfrak{D}$. To avoid confusion with other definitions we designate it by H.

$$H(z)=\sigma(z-a)^{\epsilon_0}\mathfrak{D}(z;\hat{\delta}_{l+},\hat{\delta}_{(l+1)-}),\quad\text{(VII.4)}$$

where $\sigma=\pm1$ and $\epsilon_0=0,\pm1$ as indicated in Table I, and a is some arbitrary real point in the interval $-w_0\leqslant a\leqslant w_0$. For $\epsilon_0=1$, $H(a)=0$ and a could be taken as the energy of a bound state if there were one; otherwise, $N(a)$ must vanish also. H has the following properties:

(i) $H(z)$ is analytic in the cut plane;
(ii) $H(z^*)=H^*(z)$;
(iii) $\text{Im}H(z)>0$, $\text{Im}z>0$;
(iv) for real $z=w$, $\text{Im}H(w+i0)\geqslant 0$ on P;
(v) $H(w)$ has a pole with positive residue at $w=\bar{w}_j$. $j=1,\cdots,j(\max)$;
(vi) at $w=\hat{w}_i$, $i=1,\cdots,i(\max)$, $\text{Re}H(w)$ and $\text{Im}H(w)/\text{Re}H(w)$ vanish;
(vii) if $\text{Re}\delta(w)$ tends to a finite constant for large w, then except for logarithmic factors $H(z)\sim z^q$ for large z, where $-1\leqslant q\leqslant 1$,[39]
(viii) We define $n(w)$ by $\text{Im}H(w+i0)=-\kappa(w)n(w)$; then $n(W)\leqslant 0$ on P and for large w, $n(w)=o(1)$; $n(w)$ is not necessarily square integrable. In both cases (v) and (vi), $n(w)/H(w)$ vanishes.

Note that at $\hat{w}_i$ as well as at $\bar{w}_i$ the amplitude has the value $f=i(1-\eta)/2\kappa$. Properties (i) and (iii) show that H is in fact Herglotz. We now prove (iii) for type $\tau(--)$, in which H has the form $H(z)=(z-a)\exp J(z)$, where $J(z)$ is the integral in the definition of $\mathfrak{D}$, Eq. (II.14). Setting $z=a+R\exp(i\theta)$, we have

$$\text{Im}H(z)=R[\exp\text{Re}J(z)]\sin[\theta+\text{Im}J(z)].$$

The factor $\exp\text{Re}J(z)$ is positive definite in the upper half-plane because $J(z)$ is analytic there. In order to show that $0<\theta+\text{Im}J(z)<\pi$, we parametrize θ as follows: Let $z=u+iv$, then

$$\theta=\tfrac{1}{2}\pi-\tan^{-1}\left(\frac{u-a}{v}\right)=v\int_{-a}^{\infty}dw|w+z|^{-2}.$$

Hence,

$$\theta+\text{Im}J(z)=\frac{v}{\pi}\int_{w_0}^{\infty}dw\left[\frac{-\hat{\delta}_{l+}(w)}{|w-z|^2}+\frac{\pi+\hat{\delta}_{(l+1)-}(w)}{|w+z|^2}\right]+v\int_{-a}^{w_0}dw|w+z|^{-2}.$$

The inequality (VII.3b) for $\tau(--)$ and the restriction $-a\leqslant w_0$ ensure that $\theta+\text{Im}J(z)>0$ for $v>0$. An upper bound is established by replacing each phase by its extreme value and extending the lower limit of the third integral to $-w_0$. Therefore, for $a\geqslant w_0$,

$$\theta+\text{Im}J(z)<v\int_{-\infty}^{\infty}dw|w+z|^{-2}=\pi.$$

This concludes the proof for $\tau(--)$. The other cases can be handled similarly. The remaining properties are easy to verify.

[38] It is interesting that in the case of pure elastic scattering $\hat{\delta}$ is just the argument ϕ of the partial-wave amplitude: $kf=\sin\delta\times\exp(i\delta)=|kf|\exp(i\phi)$. ϕ jumps by $\pm\pi$ whenever $\sin\delta$ vanishes.

[39] The expression "except for logarithmic factors $H(z)\sim z^q$" means precisely that $H(z)=z^q\exp\lambda(z)$ where $|\lambda(z)|<\epsilon\ln|z|$ for all z such that $|z|>R(\epsilon)$ and for any $\epsilon>0$; cf., Appendix A, Eq. (A7).

The Herglotz-Cauchy representation of H is

$$H(z)=H(0)+zA-\frac{z}{\pi}\int_P dw\frac{\kappa n(w)}{w(w-z)}+\sum_j \frac{zc_j'}{w_j(w_j-z)}, \quad \text{(VII.5)}$$

where $H(0)$, A, c_j', w_j, $j=1,\cdots,j(\max)$ are real and $A\geqslant 0$, $c_j'\geqslant 0$, and $n(w)\leqslant 0$ on P. The w_j may be restricted to P with one exception in type $\tau(++)$. The pole terms involving (w_j,c_j') include any CDD poles, but include also poles at all points where $\mathrm{Re}\delta$ goes up through a multiple of π. The number of (finite) CDD poles is $j(\max)-i(\max)$. Which of the $\bar{w}_j$ are regarded as CDD points seems to be a matter of free choice, if $i(\max)\neq 0$. This may be a significant point in connection with attempts to represent unstable particles by CDD poles.

In the formulation of the integral equation for $n(w)$ as carried out in Sec. VI, the function $H(z)$ is not useful as it stands. Among the (w_j,c_j'), only a certain number of "authentic" CDD parameters may be regarded as arbitrary; an arbitrary choice of the remaining parameters may be incompatible with a given interaction. However, a D function with a representation like (VII.5) but involving only CDD poles is available. A number $i(\max)$ of pole-zero pairs may be factored out of H. If

$$\tau(z)=\prod_{i=1}^{i(\max)}\frac{z-\bar{w}_i}{z-\hat{w}_i}, \quad \text{(VII.6)}$$

then $D=\tau H$ has the desired integral representation. This is clear after some manipulation with partial fractions. The Az term survives the transformation. This D function is the starting point of the derivation of (VI.7); see Eq. (VI.4). Of course, it is not necessarily Herglotz, and there are no restrictions on the signs of the parameters c_i beyond those implied by the requirement that the amplitude have no ghosts.

To conclude, we give some relations between the various D's. Suppose N/D is constructed as in Sec. VI with n finite CDD poles and perhaps $A\neq 0$. Then $D(w)\sim w^r$, $r\leqslant 1$, and $r=1$ if $A\neq 0$.[40] Suppose the numbers r, $i(\max)$, $j(\max)$, and p are known. The number of zeros, n_z, of D is evaluated by constructing the pole-free $\bar{D}=\rho D$, where ρ is defined at the beginning of Sec. VI. Now $\bar{D}\sim w^{r+n}$ and $\mathfrak{D}\sim w^p$ (where $\mathfrak{D}$ is defined with continuous phase δ), and the polynomial $\Phi=\bar{D}\mathfrak{D}^{-1}$ has degree $n_z=r-p+n\geqslant 0$. Thus $r-p$ is an integer. Since $r\leqslant 1$ and $p\geqslant j(\max)-i(\max)$ we have $n_z\leqslant n+i(\max)-j(\max)+1$. H differs from D by a multiplicative rational function which may be worked out from the construction (VII.4) and the relation of D to the $\mathfrak{D}$ defined with continuous phase.

[40] Here and in the following remarks the symbol $\sim$ indicates asymptotic equality "except for logarithmic factors," as explained in reference 39.

APPENDIX A

Asymptotic Behavior of $D(z)$

We examine the asymptotic behavior of $\mathfrak{D}(z;\alpha_+,\alpha_-)$ under the assumption that α_+ and α_- approach constants for large w. Each of the integrals in the exponent can be broken up as follows:

$$\int_{w_0}^{\infty} dw\frac{\alpha(w)}{w(w\mp z)}=\int_{w_0}^{c} dw\frac{\alpha(w)}{w(w\mp z)}+\alpha(r)\int_c^{\infty} dw\frac{1}{w(w\mp z)}+\int_c^{\infty} dw\frac{\alpha(w)-\alpha(r)}{w(w\mp z)}, \quad \text{(A1)}$$

where c is some constant, $w_0<c<\infty$, and $r=|z|$. The first term is $O(r^{-1})$, the second is equal to $\mp\alpha(r)z^{-1}\times\ln(1\mp zc^{-1})$, and the third term (call it J) can be treated as follows: We write

$$J=\left(\int_c^{r-\delta}+\int_{r-\delta}^{r+\delta}+\int_{r+\delta}^{\infty}\right)dw\frac{\alpha(w)-\alpha(r)}{w(w\mp z)}, \quad \text{(A2)}$$

where δ is a constant such that $0<\delta<r-c$. For any $\epsilon>0$, c can be chosen so that $|\alpha(w)-\alpha(r)|<\epsilon$ for all w, $r>c$. Then we have

$$\left|\int_{r+\delta}^{\infty}\right|\leqslant\epsilon\int_{r+\delta}^{\infty}\frac{dw}{w|w\mp z|}\leqslant\epsilon\int_{r+\delta}^{\infty}\frac{dw}{w(w-r)}=\frac{\epsilon}{r}\ln\frac{r+\delta}{\delta}, \quad \text{(A3)}$$

and

$$\left|\int_c^{r-\delta}\right|\leqslant\epsilon\int_c^{r-\delta}\frac{dw}{w(r-w)}=\frac{\epsilon}{r}\ln\frac{(r-\delta)(r-c)}{\delta c}. \quad \text{(A4)}$$

In (A3) and (A4) the inequality $|w-r|\leqslant|w\mp z|$ has been used. By reference 16, $|\alpha(w)-\alpha(r)|\leqslant A|w-r|^{\mu}$, $0<\mu\leqslant 1$. Then

$$\left|\int_{r-\delta}^{r+\delta}\right|\leqslant A\int_{r-\delta}^{r+\delta}dw\frac{|w-r|^{\mu}}{w|w\mp z|}<\frac{A}{r-\delta}\int_{r-\delta}^{r+\delta}dw|w-r|^{\mu-1}=2A\delta^{\mu}(r-\delta)^{-\mu}=O(r^{-1}). \quad \text{(A5)}$$

From (A3), (A4), and (A5) one has $|J|\leqslant 3\epsilon r^{-1}\ln r+O(r^{-1})$. With $|\exp(ab)|\leqslant\exp|ab|$, the definition (II.14) gives

$$\mathfrak{D}(z;\alpha_+,\alpha_-)=O(r^{p+\epsilon}), \quad \text{(A6)}$$

where ϵ has been redefined by a numerical factor and $\pi p=\alpha_+(\infty)+\alpha_-(\infty)$. The above proof also shows that

$$\mathfrak{D}=z^p e^{\lambda(z)}, \quad \text{(A7)}$$

where $|\lambda(z)|<\epsilon\ln r$ for all r greater than some $R(\epsilon)$ and any $\epsilon>0$.

When $\alpha(w)-\alpha(r)$ decreases rapidly enough at infinity, one gets a precise power behavior of $\mathfrak{D}(w;\alpha_+,\alpha_-)$ at large w. Write the integral as follows:

$$P\int_{w_0}^{\infty}dw'\frac{\alpha(w')}{w'(w'\pm w)}=P\int_{w_0}^{\infty}dw'\frac{\alpha(w')-\alpha(\infty)}{w'(w'\pm w)}+\alpha(\infty)P\int_{w_0}^{\infty}dw'\frac{1}{w'(w'\pm w)}. \quad \text{(A8)}$$

If $\alpha(w)-\alpha(\infty)=O(\ln^{-\alpha}w)$, $\alpha>1$, then the theorem of Appendix D shows that the first term on the right of (A8) tends to the constant $\int_{w_0}^{\infty}dw'\,[\alpha(w')-\alpha(\infty)]/w'$. Thus, $\mathfrak{D}\sim w^p$.

If α oscillates at infinity the argument leading to (A6) may still be applied if $|\alpha(w)-\alpha(r)|<\lambda$ for all w, $r>c$. Then we have

$$\mathfrak{D}(z;\alpha_+,\alpha_-)=O(r^{[\alpha+(r)+\alpha-(r)]/\pi+3\lambda/\pi}). \quad \text{(A9)}$$

APPENDIX B

Continuity Properties and Derivatives of P.V. Integrals

The following lemmas are to be proved: (A) In the "normal" case, ReD (resp. ImD) satisfies an H condition. In the CDD case, Im$D\in H$ near the CDD point w_i if $d\delta/dw\in H$ near w_i; (B) ReB has a derivative that satisfies an H condition wherever $d\eta/dw\in H$; (C) $h(w,w')=[w\,\mathrm{Re}B(w)-w'\,\mathrm{Re}B(w')](w-w')^{-1}$ satisfies an H condition in both variables wherever $d\eta/dw\in H$. The H condition in both variables means that $|h(w_1,w_2)-h(w_1',w_2')|\leqslant A|w_1-w_1'|^{\mu}+B|w_2-w_2'|^{\nu}$, $0<\mu$, $\nu\leqslant 1$.

Two theorems from Articles 19 and 20 of reference 17 will be helpful. *Theorem 1:* If $\varphi(t)$ satisfies an H condition on the interval $L=[t_1,t_2]$, then

$$\Phi(s+i0)=\int_L dt\frac{\varphi(t)}{t-(s+i0)}$$

satisfies an H condition everywhere (except possibly in arbitrarily small neighborhoods of t_1, t_2, if φ is not zero at those points). *Theorem 2:* If $\varphi(t,s)$ satisfies an H condition in both variables for $t_1\leqslant s$, $t\leqslant t_2$ then

$$\Phi(s+i0)=\int_L dt\frac{\varphi(t,s)}{t-(s+i0)}$$

satisfies an H condition for s in L, except possibly near the ends. We note also *Lemma 1:* $\Psi(s,s')=[\psi(s)-\psi(s')]\times(s-s')^{-1}$ satisfies an H condition for $s_0\leqslant s$, $s'\leqslant s_1$, if $d\psi/ds\in H$ for $s_0\leqslant s\leqslant s_1$. This follows immediately from the identity $\psi(s)-\psi(s')=(s-s')\int_0^1\psi'[s'+y(s-s')]dy$.

The first part of Lemma A follows immediately from Theorem 1, since in the normal case $D=\Phi\mathfrak{D}$, where Φ is a polynomial. In the CDD case the same argument applies to ImD except near the points w_i. Near w_i, only the factor $\sin\delta(w)/(w-w_i)$ of ImD is in question. Let $\psi(w)=\sin\delta(w)$ in Lemma 1. Since $\psi(w_i)=0$, it follows that $\sin\delta(w)/(w-w_i)\in H$ near w_i if $d\delta/dw\in H$ near w_i.

To prove Lemma B it is sufficient to consider only the term Ref^I in ReB, since f^U is analytic in a region including P. Compute $d(\mathrm{Re}f^I)/dw$ from

$$\mathrm{Re}f^I(w)=\int_I dw'\frac{w'\varphi(w')-w\varphi(w)}{w'-w}+w\varphi(w)P\int_I dw'\frac{1}{w'(w'-w)},$$

where $\varphi=(1-\eta)(2\pi\kappa)^{-1}$.

$$\frac{d(\mathrm{Re}f^I)}{dw}=\int_I dw'\frac{1}{w'-w}\times\left[\frac{w'\varphi(w')-w\varphi(w)}{w'-w}-\frac{d}{dw}[w\varphi(w)]\right]-\frac{d}{dw}\left[w\varphi(w)P\int_I dw'\frac{1}{w'(w'-w)}\right]. \quad \text{(B1)}$$

From (B.1), Lemma 1, and Theorems 1 and 2 we conclude that $d(\mathrm{Re}f^I)/dw\in H$ if $d\eta/dw\in H$. Lemma C follows from Lemmas B and 1.

APPENDIX C

Changing the Order of Repeated P.V. Integrations

The Poincaré-Bertrand formula (III.6) is proved in Sec. 23 of reference 17 for the case in which L is a finite interval $[x_0,x_1]$. The weight function $\varphi(x,y)$ is assumed to satisfy an H condition in both x and y, and the point t is not to coincide with either of the end points x_0, x_1. Here we show that (III.6) also holds when L is the infinite region $[x_0,\infty]$ and φ satisfies the conditions of our problem.

The proof of reference 23 can be extended trivially to the case in which the x and y integrations are over $L=[x_0,X]$ and $M=[x_0,Y]$, respectively, and the point t is common to L and M. Now suppose that $I_y=P\int_{x_0}^{\infty}\varphi(x,y)(x-y)^{-1}dy$ converges uniformly for $x_0\leqslant x\leqslant X$. Then it follows that

$$P\int_{x_0}^{X}\frac{dx}{x-t}P\int_{x_0}^{\infty}\frac{\varphi(x,y)dy}{y-x}=-\pi^2\varphi(t,t)+\int_{x_0}^{\infty}dy\,P\int_{x_0}^{X}\frac{\varphi(x,y)dx}{(x-t)(y-x)}, \quad \text{(C1)}$$

provided the integral on the right side converges. The limit $X\to\infty$ gives the desired result if the limit can be taken inside the y integral. The latter operation is permissible if

$$\lim_{X\to\infty}\int_{x_0}^{\infty}dy\,P\int_{X}^{\infty}\frac{\varphi(x,y)dx}{(x-t)(y-x)}=0. \quad \text{(C2)}$$

We now wish to show that the various conditions mentioned are all fulfilled in the problem of Sec. III.

In Sec. III, $\varphi(x,y)=g(x)h(y)$. In fact,

$$\varphi(w,w')=\frac{1-\eta(w)}{\kappa(w)}\frac{\operatorname{Im}D(w')}{w'}.$$

Therefore, $g(x)=O(x^{-1})\geqslant 0$, $h(x)=O(x^{-\epsilon})$, $\epsilon>0$. Also g, $\varphi\in H$. The uniform convergence of I_y and the convergence of the integral on the right side of (C1) follow immediately. To prove (C2) write $\int_{x_0}^{\infty}=\int_{x_0}^{X}+\int_X^{\infty}=I_1+I_2$. We first examine I_1. Now for y, $t<X$,

$$\left|\int_X^{\infty}dx\frac{g(x)}{(x-t)(y-x)}\right|\leqslant M\int_X^{\infty}dx\frac{1}{x(x-y)}=\frac{M}{y}\ln\frac{X}{X-y}.$$

Hence,

$$|I_1|\leqslant M\int_{x_0}^{X}dx\frac{1}{x^{1+\epsilon}}\ln\frac{X}{X-x}$$
$$=\int_{x_0}^{X(1-\delta)}+\int_{X(1-\delta)}^{X}=J_1+J_2,$$

where $0<\delta<1$ and $x_0<X(1-\delta)$. After a partial integration and use of an upper bound for a factor of the integrand we find $J_1=O(X^{-\epsilon})$. Moreover, $J_2=O(X^{-\epsilon}\ln X)$. Thus, $I_1\rightarrow 0$. To treat I_2 write

$$|I_2|\leqslant M\int_X^{\infty}dy\frac{1}{y^{\epsilon}(y-t)}\times\left|P\int_X^{\infty}dx\, g(x)\left(\frac{1}{x-t}-\frac{1}{x-y}\right)\right|. \quad \text{(C3)}$$

Call the two terms in the absolute value sign K_1 and K_2. Since $g(x)\geqslant 0$, $|K_1|\leqslant\int_{X_0}^{\infty}dx\, g(x)(x-t)^{-1}=M(t)$ for all $X>X_0>t$. In K_2 we subtract out the pole:

$$-K_2=\int_X^{\infty}dx\frac{xg(x)-yg(y)}{x(x-y)}+yg(y)P\int_X^{\infty}dx\frac{1}{x(x-y)}=L_1+L_2.$$

Since $g\in H$ we have

$$|L_1|\leqslant M\int_X^{\infty}dx\, x^{-1}|x-y|^{-1+\mu}=O(1),\quad 0<\mu\leqslant 1.$$

After evaluation of L_2 and some obvious estimates, (C3) shows that $I_2\rightarrow 0$.

APPENDIX D

Asymptotic Behavior of P.V. Integrals

Theorem: If $\varphi(x)=O(x^{-1}\ln^{-\alpha}x)$, $\alpha>0$, and $\varphi'(x)$ exists and is continuous, then

$$tP\int_{x_0}^{\infty}\frac{\varphi(x)dx}{t-x}=\int_{x_0}^{t}\varphi(x)dx+O(\ln^{-\alpha}t), \quad \text{(D1)}$$

where $x_0>0$. For the proof we suppress the symbol P and decompose the integral as follows.

$$t\int_{x_0}^{\infty}\frac{\varphi(x)dx}{t-x}=\int_{x_0}^{t(1+\epsilon)}\varphi(x)dx+\int_{x_0}^{t(1-\epsilon)}\frac{x\varphi(x)}{t-x}dx+\int_{t(1-\epsilon)}^{t(1+\epsilon)}\frac{x\varphi(x)}{t-x}dx+t\int_{t(1+\epsilon)}^{\infty}\frac{\varphi(x)dx}{t-x}=I_1+I_2+I_3+I_4,$$

where $0<\epsilon<1$ and $x_0<t(1-\epsilon)$. Since

$$|I_2|\leqslant Mt^{-1}\int_{x_0}^{t(1-\epsilon)}\ln^{-\alpha}x dx=Mt^{-1}J_2,$$

we can apply l'Hôpital's rule to $J_2/t\ln^{-\alpha}t$ to show that $I_2=O(\ln^{-\alpha}t)$. For I_4 we have

$$|I_4|\leqslant Mt\ln^{-\alpha}[t(1+\epsilon)]\int_{t(1+\epsilon)}^{\infty}x^{-1}(x-t)^{-1}dx=O(\ln^{-\alpha}t).$$

I_3 can be written in the form

$$I_3=\int_{t(1-\epsilon)}^{t(1+\epsilon)}\frac{x\varphi(x)-t\varphi(t)}{t-x}dx.$$

But $[x\varphi(x)-t\varphi(t)](x-t)^{-1}=[x\varphi(x)]_{x=\xi}'$, where ξ is between x and t. By reference 26, $[x\varphi(x)]'=O(x^{-1}\ln^{-\alpha}x)$. It follows that $I_3=O(\ln^{-\alpha}t)$. Finally we note that

$$I_1=\int_{x_0}^{t}\varphi(x)dx+O(\ln^{-\alpha}t),$$

and the proof is complete. With the alternative hypothesis $\varphi(x)=O(x^{-\alpha})$, $\alpha>0$, $\varphi\in C^1$, similar reasoning shows that the integral on the left of (D1) is $O(t^{1-\alpha})$. Finally, if $\varphi(x)=O(\ln^{-\alpha}x)$, $\alpha>1$, $\varphi\in C^1$, then the integral is $O(t\ln^{1-\alpha}t)$. If the denominator $t-x$ in (D1) is replaced by $t+x$, a similar theorem holds without any assumption about the derivative of φ. In that case one also has a related result due to Hardy and Littlewood[41]: If

$$G(t)=\int_{x_0}^{t}x\varphi(x)dx=o(t),$$

then

$$t\int_{x_0}^{\infty}\frac{\varphi(x)}{t+x}dx-\int_{x_0}^{t}\varphi(x)dx=o(1).$$

[41] G. H. Hardy and J. E. Littlewood, Proc. London Math. Soc. **30**, 23 (1930). See also, reference 42 and D. Amati, M. Fierz, and V. Glaser, Phys. Rev. Letters **4**, 89 (1960).

The proof of this theorem given by Widder[42] can be adapted in a straightforward manner to fit the principal value case. One finds that $G(t)=o(t)$ implies

$$tP\int_{x_0}^{\infty}\frac{\varphi(x)}{t-x}dx-\int_{x_0}^{t(1+\epsilon)}\varphi(x)dx -P\int_{s(1-\epsilon)}^{s(1+\epsilon)}\frac{x\varphi(x)}{t-x}dx=o(1),$$

where $\epsilon>0$ is arbitrarily small.

APPENDIX E

Evaluation of the Amplitude Near $w=\pm(M-m)$

We use crossing symmetry to evaluate the amplitude $f_{l\pm}(w)$, rather than just its discontinuity, near $w=\pm(M-m)$. The essential point can be seen in terms of the Legendre projection $A_l^{(\pm)}(s)$ of the invariant amplitude $A^{(\pm)}(s,\bar{s},t)=\pm A^{(\pm)}(\bar{s},s,t)$.[43] This is

$$A_l^{(\pm)}(s)=\pm\int_{-1}^{1}dx\,P_l(x) \times A^{(\pm)}(2M^2+2m^2-s+2k^2(1-x),\,s,\,-2k^2(1-x)),\quad\text{(E1)}$$

where in the barycentric system of the crossed channel $\bar{s}=2M^2+2m^2-s+2k^2(1-x)$ is the square of the total energy and $\bar{z}=1+t(2\bar{k}^2)^{-1}$ is the cosine of the scattering angle. The magnitude of the barycentric three-momentum $\bar{k}$ is expressed in terms of $\bar{s}$ in the same way that k is expressed in terms of s. For s in some domain in the complex s plane, it can be shown that $\bar{s}\geqslant(M+m)^2$ and that $\bar{z}$ lies within the Lehmann ellipse for all x in the interval $-1\leqslant x\leqslant 1$. We need only the more limited result that for real s in $0\leqslant s\leqslant(M-m)^2$, $\bar{s}\geqslant(M+m)^2$, and $-1\leqslant\bar{z}\leqslant 1$ for all relevant x. It is, therefore, permissible to expand $A^{(\pm)}(\bar{s},s,t)$ in terms of partial waves of the $\bar{s}$ channel.

We set $s=(M-m)^2-\delta s$ and keep only the leading terms in δs. Then

$$\bar{s}=(M+m)^2+\delta s[1+2mM(M-m)^{-2}(1-x)]+O[(\delta s)^2],$$

and

$$\bar{k}=mM(M^2-m^2)^{-1}(2\delta s)^{1/2} \times[(M^2+m^2)(2Mm)^{-1}-x]^{1/2}+O(\delta s).$$

We take into account unitarity in the $\bar{s}$ channel; for a state of definite isotopic spin $I=\frac{1}{2},\frac{3}{2}$, we have

$$f_{l\pm}^{(I)}(\bar{s})=a(I,l\pm)\bar{k}^{2l}(1-ia(I,l\pm)\bar{k}^{2l+1})^{-1} \approx a(I,l\pm)\bar{k}^{2l}+ia^2(I,l\pm)\bar{k}^{4l+1},$$

for small $\bar{k}$. To lowest order in δs, the amplitudes $A^{(\pm)}$ and $B^{(\pm)}(s,\bar{s},t)=\mp B^{(\pm)}(\bar{s},s,t)$ are given by

$$A^{(\pm)}(\bar{s},s,t)=4\pi(2M+m)(2M)^{-1}(a(\pm)+ib(\pm)\bar{k}) -8\pi mMc(\pm)\quad\text{(E2a)}$$

and

$$B^{(\pm)}(\bar{s},s,t)=4\pi(2M)^{-1}(a(\pm)+ib(\pm)\bar{k}) +8\pi Mc(\pm),\quad\text{(E2b)}$$

plus terms of order δs, where

$$\begin{pmatrix}a(+)\\a(-)\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1&2\\1&-1\end{pmatrix}\begin{pmatrix}a(\frac{1}{2},0+)\\a(\frac{3}{2},0+)\end{pmatrix},$$

$$\begin{pmatrix}b(+)\\b(-)\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1&2\\1&-1\end{pmatrix}\begin{pmatrix}a^2(\frac{1}{2},0+)\\a^2(\frac{3}{2},0+)\end{pmatrix},$$

and $c(\pm)$ are related to the four p-wave scattering lengths $a(I,1\pm)$. To this order $\bar{z}$ does not enter. The final form is best given in terms of $f_{l\pm}(w)$.

Let $w=M-m-\delta w$, then $s=(M-m)^2-\delta s+O[(\delta s)^2]$ (for small $\delta w>0$) where $\delta s=2(M-m)\delta w$. Now $f_{l\pm}(w)$ is expanded in terms of $A_l^{(\pm)}(s)$ and $B_l^{(\pm)}(s)$ by

$$f_{l\pm}(M-m-\delta w) =2M[16\pi(M-m)]^{-1}\{A_l[(M-m)^2-\delta s] -mB_l[(M-m)^2-\delta s]\}+O(\delta w).\quad\text{(E3)}$$

It is remarkable that $f_{l+}(M-m-\delta w)=f_{l-}(M-m-\delta w)+O(\delta w)$. Inserting (E1), a corresponding relation for $B_l^{(\pm)}$ (but with the $\pm$ inverted), and (E2) into (E3), we have[44]

$$f_{l+}^{(\pm)}(M-m-\delta w)=\pm\frac{M+m}{M-m}a(\pm)\delta_{l0} \pm ib(\pm)\xi_l\left(\frac{mM}{M-m}\right)\left(\frac{\delta w}{M-m}\right)^{1/2}+O(\delta w),\quad\text{(E4)}$$

where

$$\xi_l=\int_{-1}^{1}dx\,P_l(x)[(M^2+m^2)(2mM)^{-1}-x]^{1/2}.$$

The value near $w=-M+m$ is given by the MacDowell symmetry. It is very interesting that the behavior for every l is given in terms of the S-wave scattering lengths, alone.

It is easy to verify that the Mandelstam representation implies the behavior k^{2l} and k^{2l+2} at the physical thresholds w_0 and $-w_0$. Expand Mandelstam denominators containing $\cos\theta$ in terminating Taylor series. A partial-wave projection then gives the result, after a straightforward justification of the changed order of integrations. This procedure fails for $s=(M-m)^2$ because the Mandelstam denominator $\bar{s}'-[2M^2+2m^2-s+2k^2(1-\cos\theta)]$ vanishes for $\bar{s}'=(M+m)^2$.

[42] D. V. Widder, *The Laplace Transform* (Princeton University Press, Princeton, New Jersey, 1941), Chap. 5, Sec. 5, Lemma 5.

[43] The discussion of this Appendix is restricted to pion-nucleon scattering. The notation is that of Frazer and Fulco, reference 13

[44] (E4) does not agree with some unproved statements about the behavior near $\pm(M-m)$ in reference 13.